Copulas: A Tool For Modelling Dependence In Finance

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Joint work with a lot of people (see ref. further)

The Working Paper "Copulas For Finance" is available on the web site: *http://www.gloriamundi.org/var/wps.html*

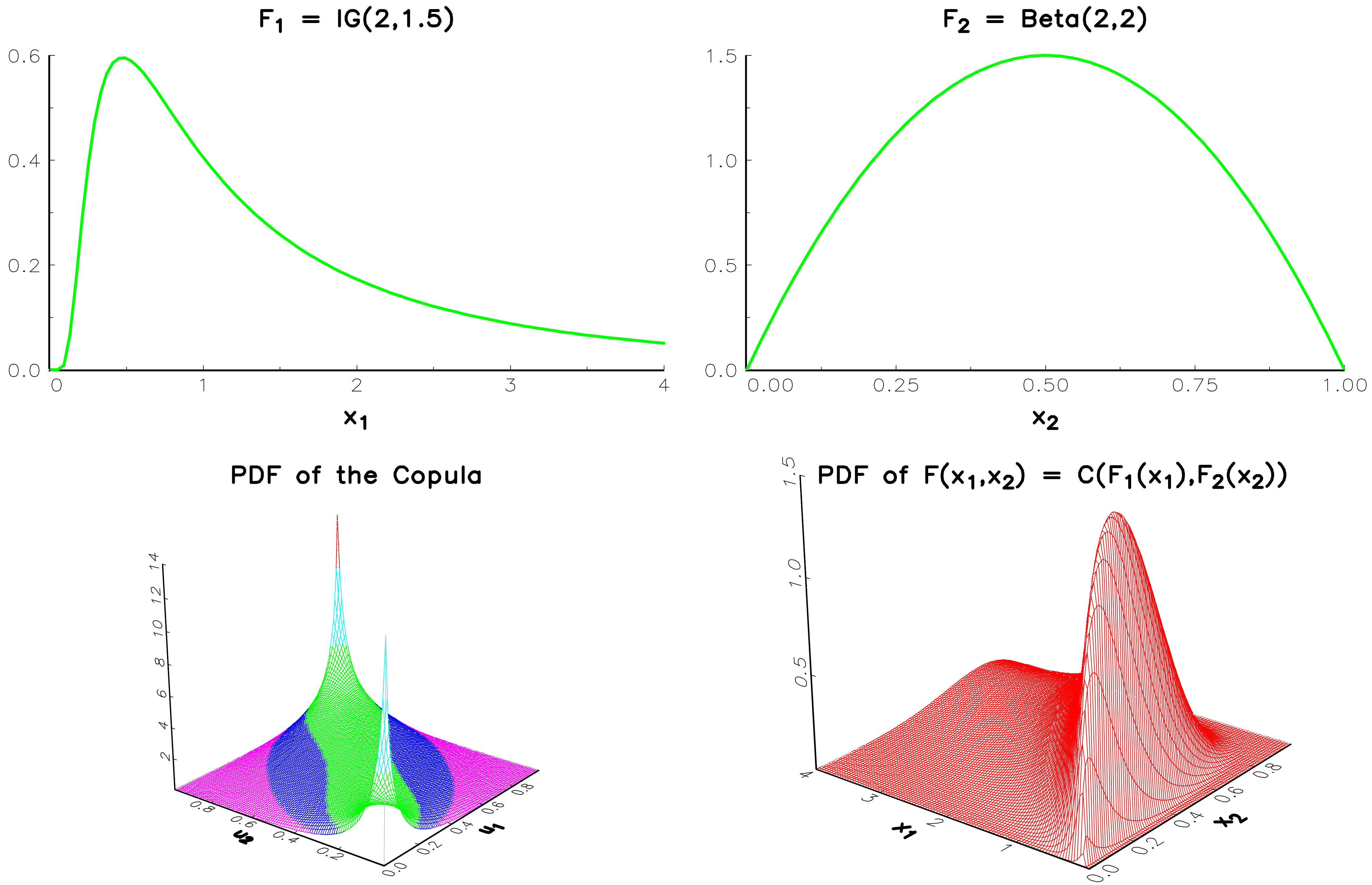
1 Introduction

Definition 1 *A copula function* $\mathbf{C}$ *is a multivariate* **uniform** *distribution (a multivariate distribution with uniform margins).*

Theorem 1 *Let* $\mathbf{F}_1, \ldots, \mathbf{F}_N$ *be* N *univariate distributions. It comes that* $\mathbf{C}\left(\mathbf{F}_1\left(x_1\right), \ldots, \mathbf{F}_n\left(x_n\right), \ldots, \mathbf{F}_N\left(x_N\right)\right)$ *defines a multivariate distributions* $\mathbf{F}$ *with margins* $\mathbf{F}_1, \ldots, \mathbf{F}_N$ *(because the integral transforms are uniform distributions).*

$\Rightarrow$ $\mathbf{F}$ belongs to the Fréchet class $\mathcal{F}\left(\mathbf{F}_1, \ldots, \mathbf{F}_N\right)$ — $\mathbf{F}$ is a distribution with given marginals.

$\Rightarrow$ Copulas are also a general tool to construct multivariate distributions, and so multivariate statistical models — see for example Song [2000].



Bivariate distribution with given marginals

2 The dependence function

- Canonical representation
- Concordance order
- Measure of dependence

From 1958 to 1976, virtually all the results concerning copulas were obtained in connection with the study and development of the theory of probabilistic metric spaces (Schweizer [1991]).

$\Rightarrow$ Schweizer and Wolff [1976] = connection with rank statistics (see also Deheuvels [1979b]).

2.1 Canonical representation

Theorem 2 (Sklar's theorem) *Let* $\mathbf{F}$ *be a* N*-dimensional distribution function with* **continuous** *margins* $\mathbf{F}_1, \ldots, \mathbf{F}_N$*. Then* $\mathbf{F}$ *has a unique copula representation*

$$\mathbf{F}(x_1, \ldots, x_N) = \mathbf{C}(\mathbf{F}_1(x_1), \ldots, \mathbf{F}_N(x_N))$$

$\Rightarrow$ Copulas are also a powerful tool, because the modelling problem could be decomposed into two steps:

- Identification of the marginal distributions;
- Defining the appropriate copula function.

In terms of the density, we have the following canonical representation $f(x_1, \ldots, x_N) = c(\mathbf{F}_1(x_1), \ldots, \mathbf{F}_N(x_N)) \times \prod_{n=1}^{N} f_n(x_n)$.

The copula function of **random variables** $(X_1, \ldots, X_N)$ is **invariant** under strictly increasing transformations $(\partial_x h_n(x) > 0)$:

$$\mathbf{C}_{X_1,\ldots,X_N} = \mathbf{C}_{h_1(X_1),\ldots,h_N(X_N)}$$

> *... the copula is invariant while the margins may be changed at will, it follows that is precisely the copula which captures those properties of the joint distribution which are invariant under a.s. strickly increasing transformations* (Schweizer and Wolff [1981]).

$\Rightarrow$ **Copula = dependence function of random variables.**

This property was <u>already</u> etablished by Deheuvels [1978,1979a].

2.2 Examples

For the Normal copula, We have

$$\mathbf{C}\left(u_1,\ldots,u_N;\rho\right)=\Phi_\rho\left(\Phi^{-1}\left(u_1\right),\ldots,\Phi^{-1}\left(u_N\right)\right)$$

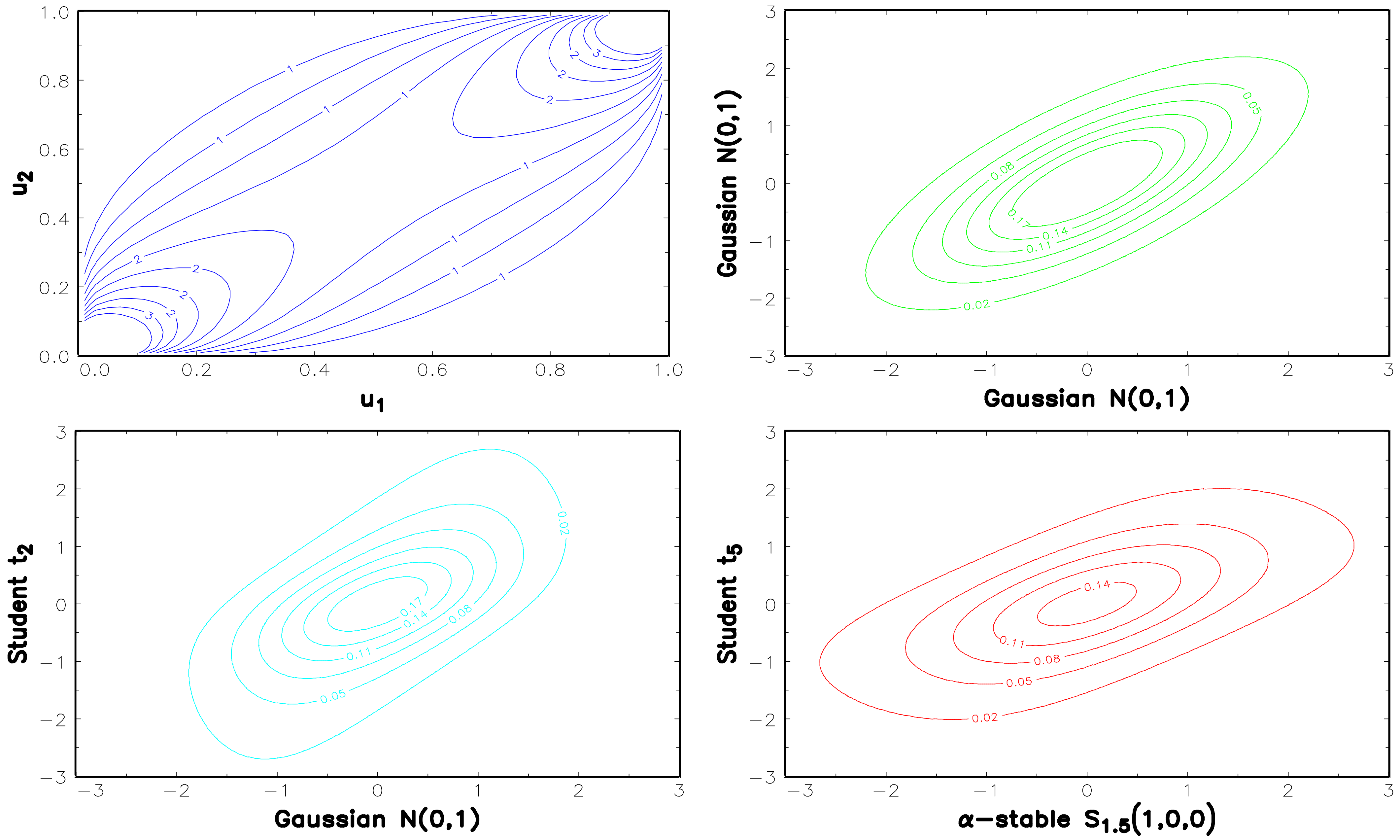
and

$$c\left(u_1,\ldots,u_N;\rho\right)=\frac{1}{\left|\rho\right|^{\frac{1}{2}}}\exp\left(-\frac{1}{2}\varsigma^\top\left(\rho^{-1}-\mathbb{I}\right)\varsigma\right)$$

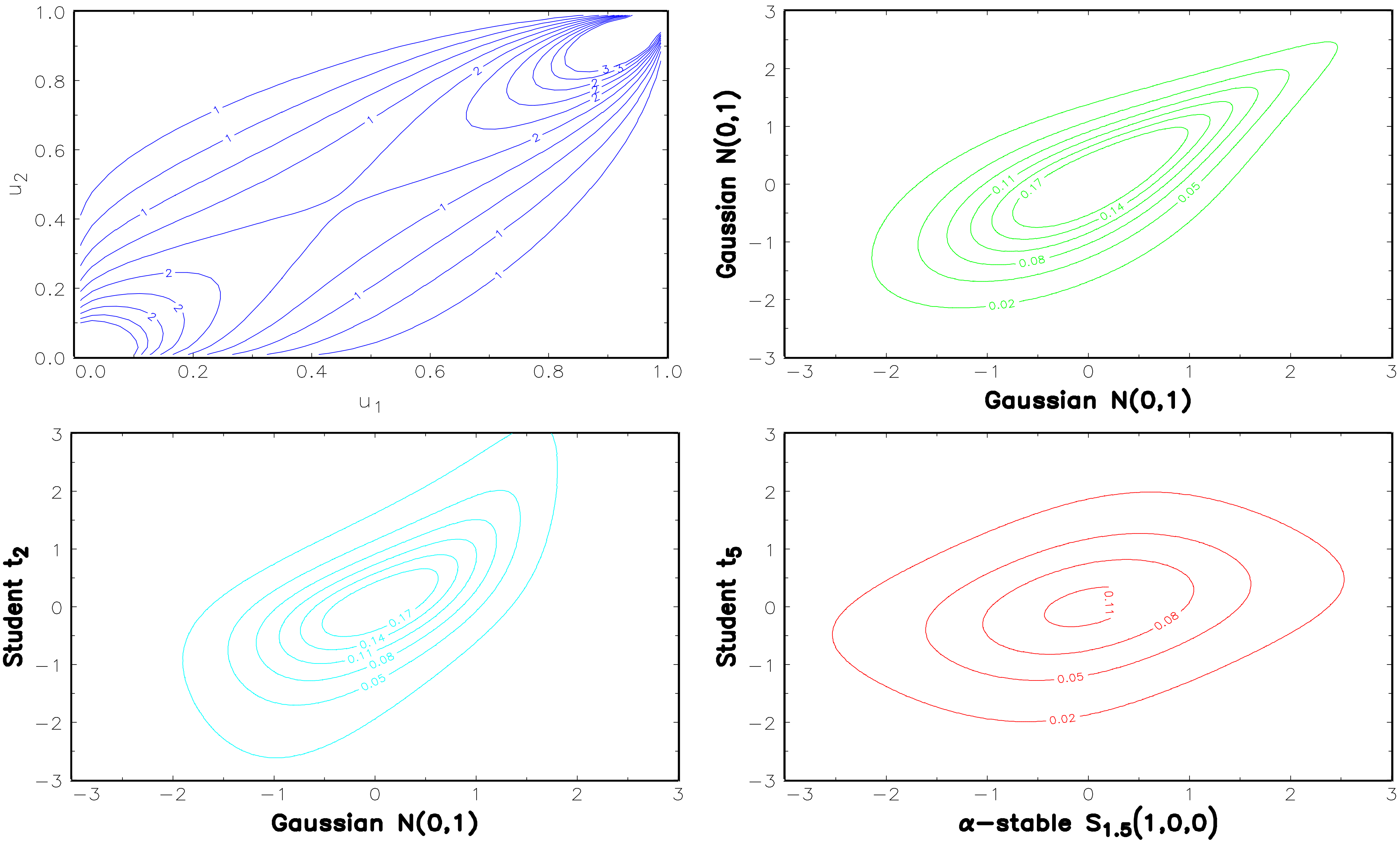
For the Gumbel copula, We have

$$\mathbf{C}\left(u_1,u_2\right)=\exp\left(-\left(\left(-\ln u_1\right)^\delta+\left(-\ln u_2\right)^\delta\right)^{\frac{1}{\delta}}\right)$$

Other copulas: Archimedean, Plackett, Frank, Student, Clayton, etc.



Contours of density for Normal copula (Kendall's tau = 0.5)



Contours of density for Gumbel copula (Kendall's tau = 0.5)

2.3 Concordance order

The copula $\mathbf{C}_1$ is **smaller** than the copula $\mathbf{C}_2$ ($\mathbf{C}_1 \prec \mathbf{C}_2$) if

$$\forall\,(u_1,\ldots,u_N) \in \mathbf{I}^N, \quad \mathbf{C}_1(u_1,\ldots,u_N) \leq \mathbf{C}_2(u_1,\ldots,u_N)$$

⇒ The lower and upper Fréchet bounds $\mathbf{C}^-$ and $\mathbf{C}^+$ are

$$\begin{aligned}
\mathbf{C}^-(u_1,\ldots,u_N) &= \max\left(\sum_{n=1}^{N} u_n - N + 1, 0\right) \\
\mathbf{C}^+(u_1,\ldots,u_N) &= \min(u_1,\ldots,u_N)
\end{aligned}$$

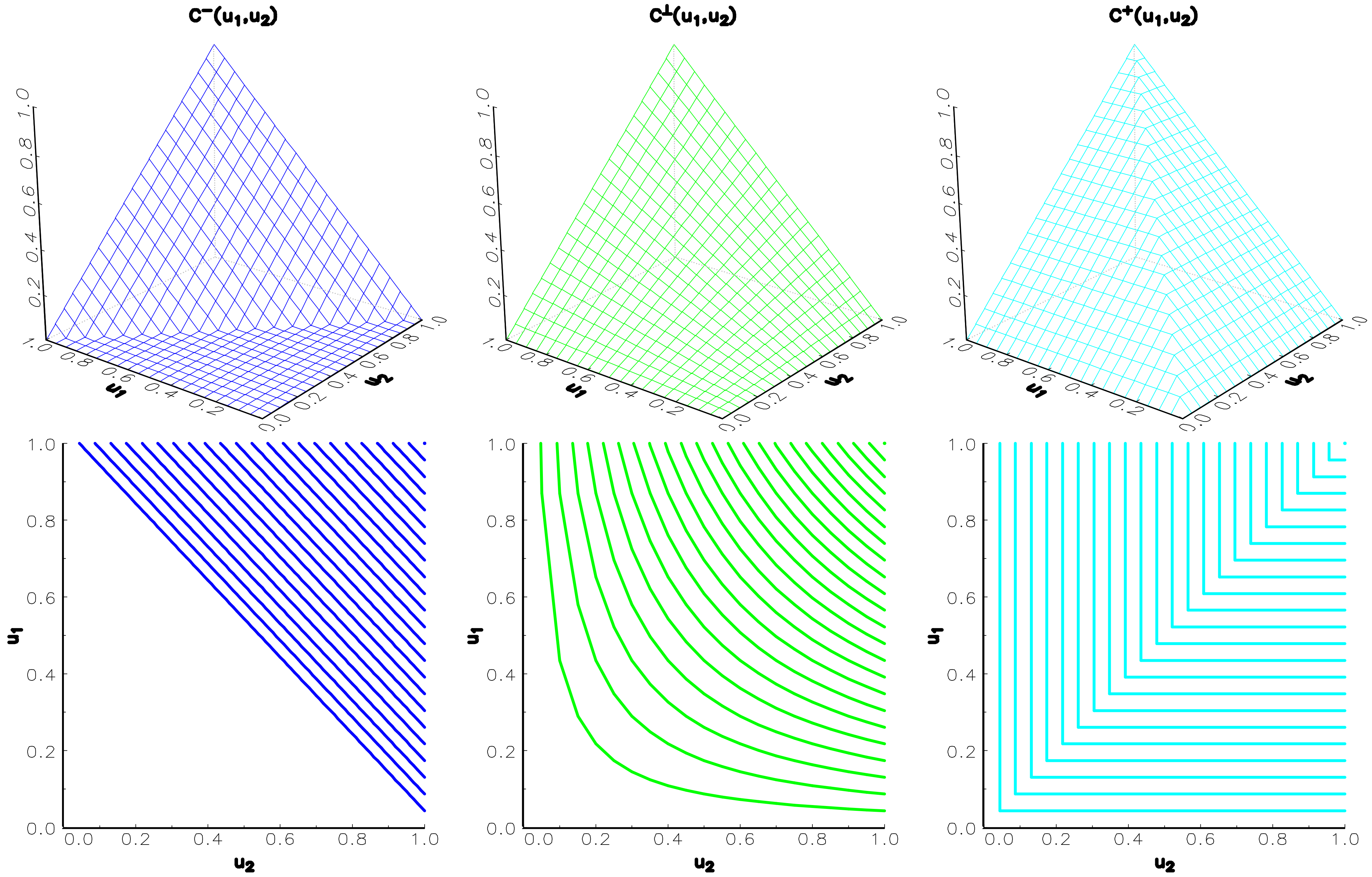
We can show that the following order holds for any copula $\mathbf{C}$:

$$\mathbf{C}^- \prec \mathbf{C} \prec \mathbf{C}^+$$

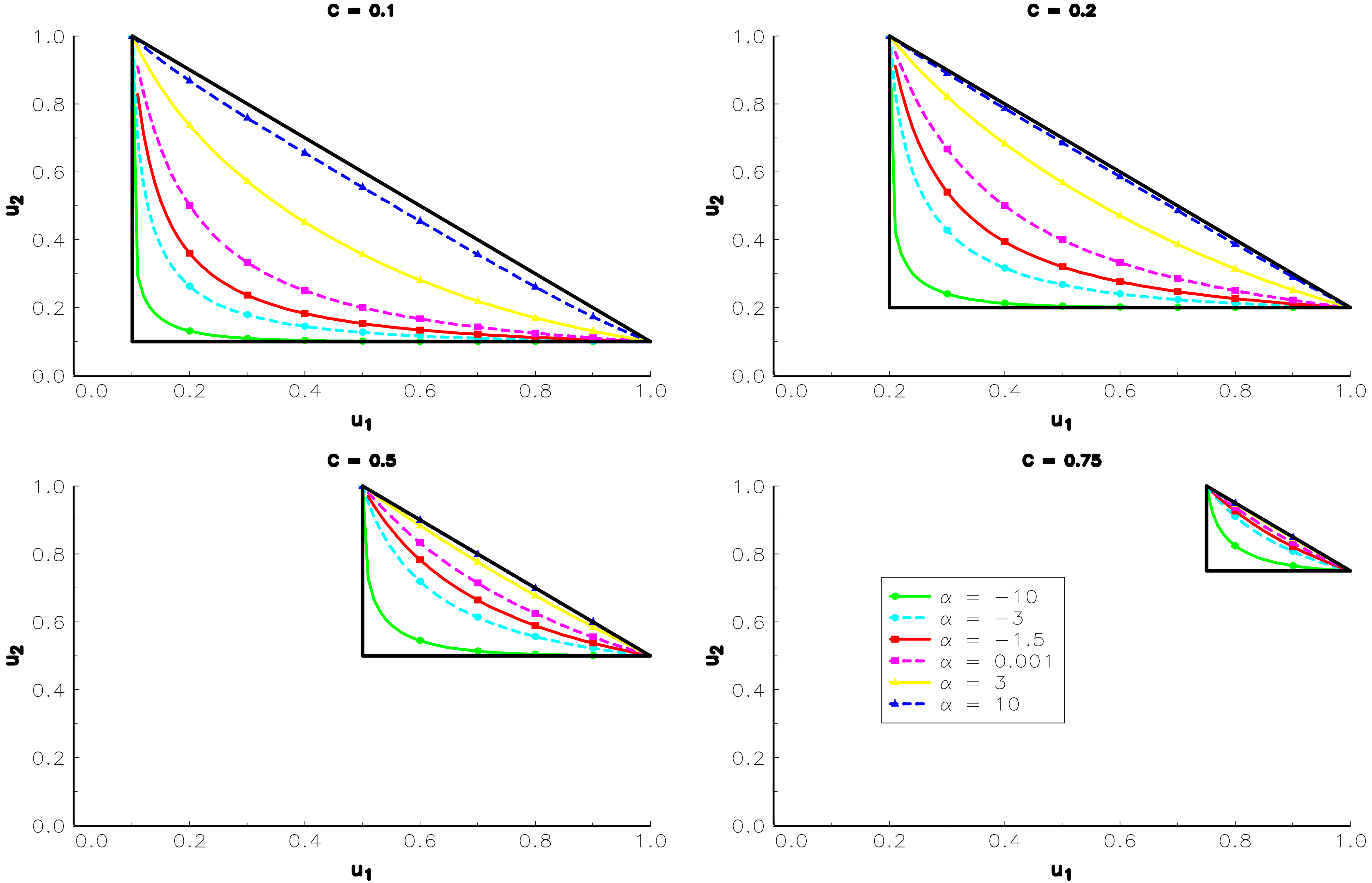
⇒ The minimal and maximal distributions of the Fréchet class $\mathcal{F}(\mathbf{F}_1,\mathbf{F}_2)$ are then $\mathbf{C}^-(\mathbf{F}_1(x_1),\mathbf{F}_2(x_2))$ and $\mathbf{C}^+(\mathbf{F}_1(x_1),\mathbf{F}_2(x_2))$.

Example of the bivariate Normal copula ($\mathbf{C}^\perp(u_1,u_2) = u_1 u_2$):

$$\mathbf{C}^- = \mathbf{C}_{-1} \prec \mathbf{C}_{\rho<0} \prec \mathbf{C}_0 = \mathbf{C}^\perp \prec \mathbf{C}_{\rho>0} \prec \mathbf{C}_1 = \mathbf{C}^+$$



Lower Frechet, product and upper Frechet copulas



Level curves of the Frank copula

Mikusiński, Sherwood and Taylor [1991] give the following interpretation of the three copulas $\mathbf{C}^{-}$, $\mathbf{C}^{\perp}$ and $\mathbf{C}^{+}$:

- Two random variables X_1 and X_2 are **countermonotonic** — or $\mathbf{C} = \mathbf{C}^{-}$ — if there exists a r.v. X such that $X_1 = f_1(X)$ and $X_2 = f_2(X)$ with f_1 non-increasing and f_2 non-decreasing;
- Two random variables X_1 and X_2 are **independent** if the dependence structure is the product copula $\mathbf{C}^{\perp}$;
- Two random variables X_1 and X_2 are **comonotonic** — or $\mathbf{C} = \mathbf{C}^{+}$ — if there exists a random variable X such that $X_1 = f_1(X)$ and $X_2 = f_2(X)$ where the functions f_1 and f_2 are non-decreasing;

2.4 Measures of association or dependence

If κ is a *measure of concordance*, it satisfies the properties: $-1 \leq \kappa_{\mathbf{C}} \leq 1$; $\mathbf{C}_1 \prec \mathbf{C}_2 \Rightarrow \kappa_{\mathbf{C}_1} \leq \kappa_{\mathbf{C}_2}$; etc.

Schweizer and Wolff [1981] show that Kendall's tau and Spearman's rho can be (re)formulated in terms of copulas

$$\begin{aligned} \tau &= 4 \iint_{\mathbf{I}^2} \mathbf{C}(u_1, u_2)\, \mathrm{d}\mathbf{C}(u_1, u_2) - 1 \\ \varrho &= 12 \iint_{\mathbf{I}^2} u_1 u_2 \, \mathrm{d}\mathbf{C}(u_1, u_2) - 3 \end{aligned}$$

$\Rightarrow$ The linear (or Pearson) correlation is not a measure of dependence.

2.5 Some misinterpretations of the correlation

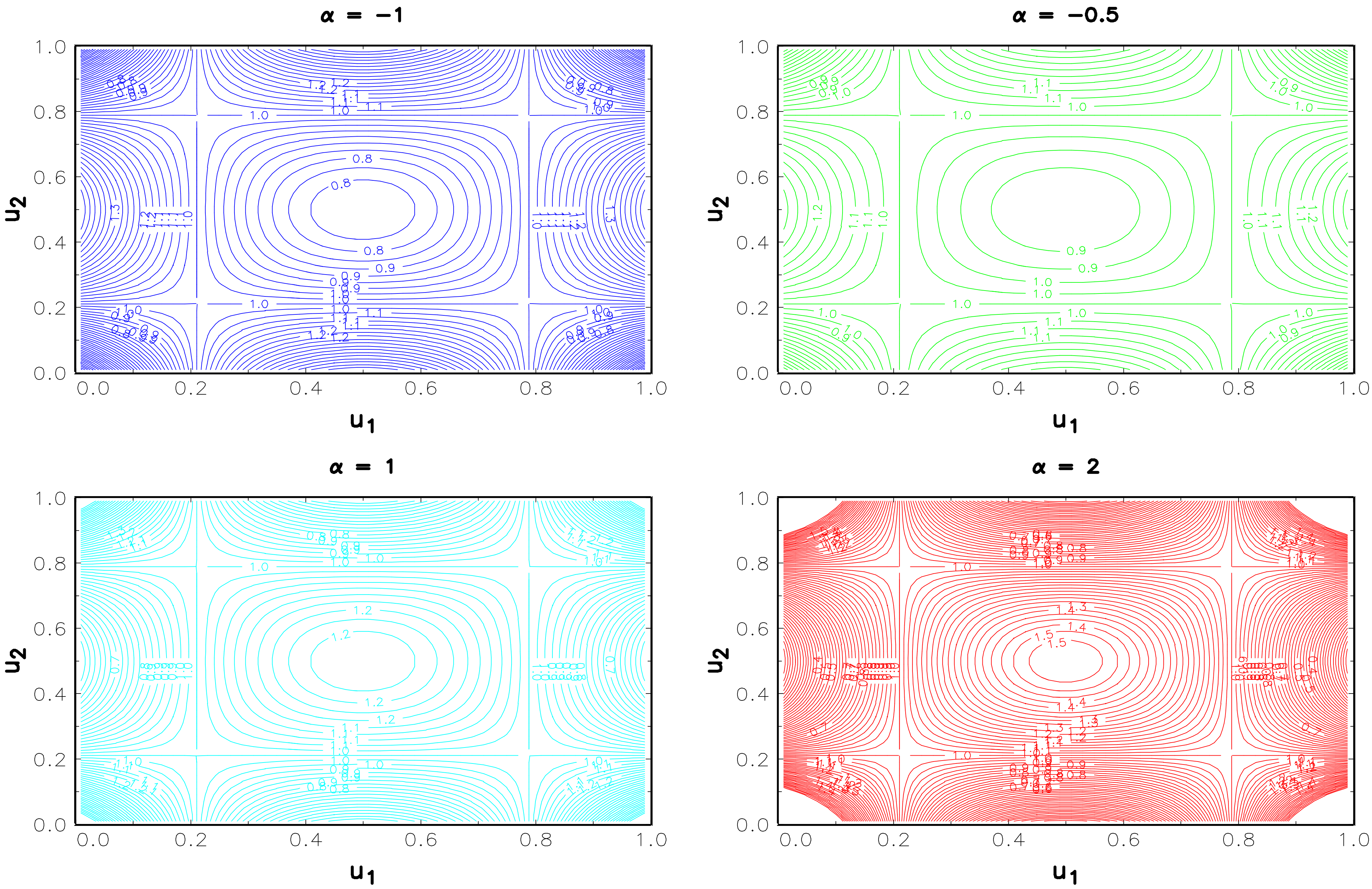
The following statements are **false**:

1. X_1 and X_2 are independent **if and only if** $\rho(X_1, X_2) = 0$;
2. For given margins, the permissible range of $\rho(X_1, X_2)$ is $[-1, 1]$;
3. $\rho(X_1, X_2) = 0$ means that there are no relationship between X_1 and X_2.

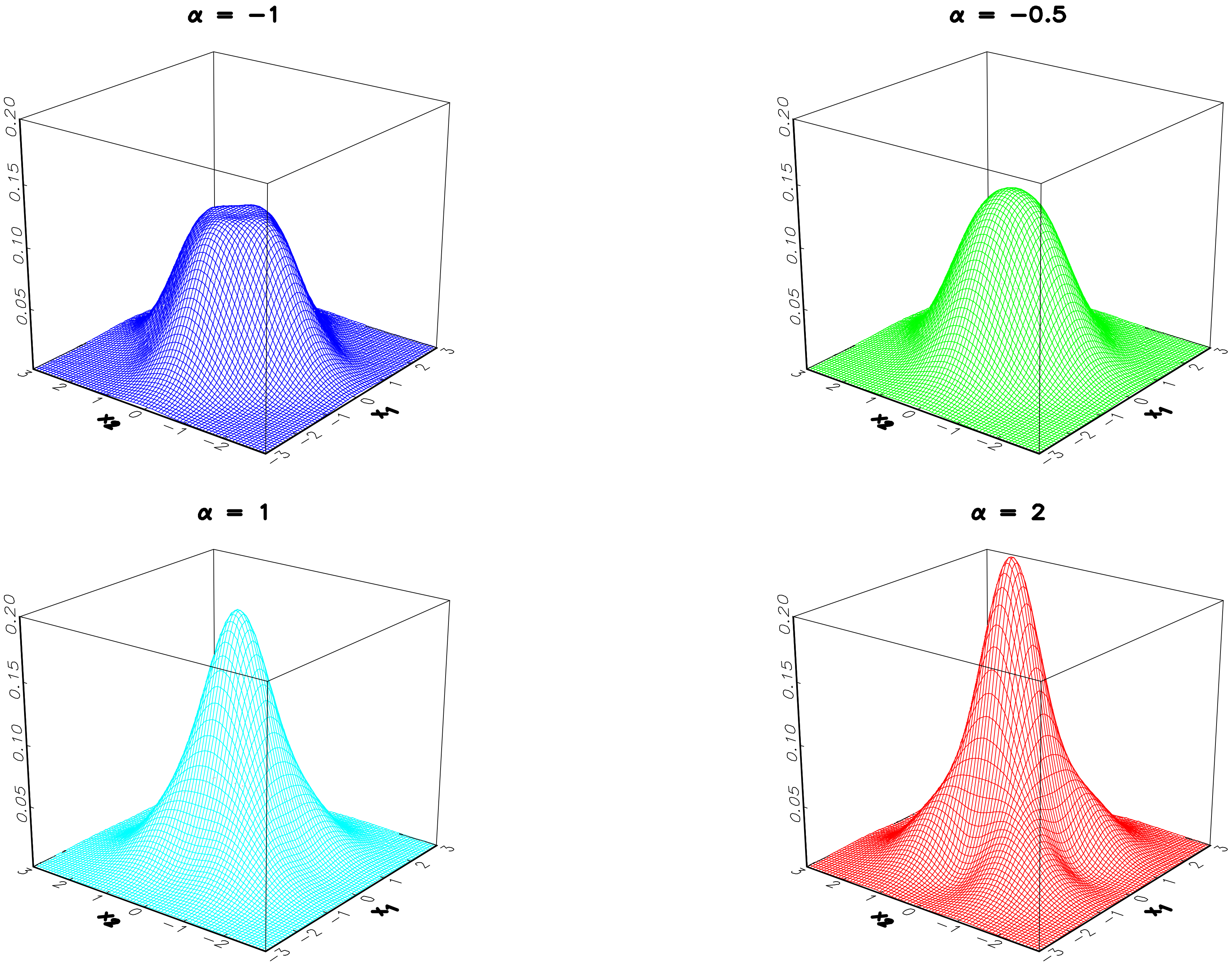
- We consider the cubic copula of Durrleman, Nikeghbali and Roncalli [2000b]

$$\mathbf{C}(u_1, u_2) = u_1 u_2 + \alpha [u_1(u_1 - 1)(2u_1 - 1)] [u_2(u_2 - 1)(2u_2 - 1)]$$

with $\alpha \in [-1, 2]$. If the margins $\mathbf{F}_1$ and $\mathbf{F}_2$ are continous and symmetric, the Pearson correlation is zero. Moreover, if $\alpha \neq 0$, the random variables X_1 and X_2 are not independent.



Contours of density for the cubic copula



PDF of the cubic copula with Gaussian margins

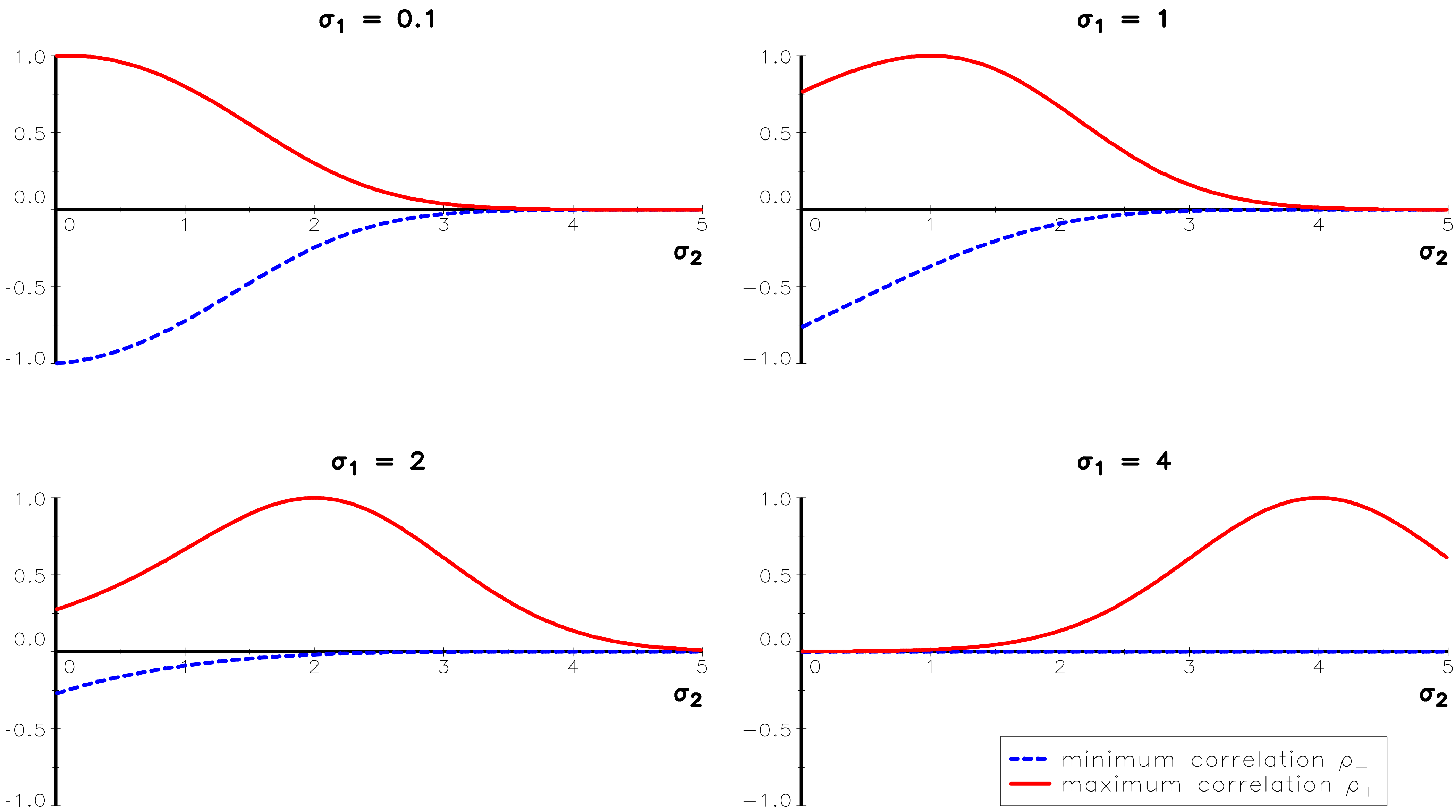
- Wang [1997] shows that the min. and max. correlations of $X_1 \sim \mathcal{LN}(\mu_1, \sigma_1)$ and $X_2 \sim \mathcal{LN}(\mu_2, \sigma_2)$ are

$$\rho_- = \frac{e^{-\sigma_1 \sigma_2} - 1}{\left(e^{\sigma_1^2} - 1\right)^{\frac{1}{2}} \left(e^{\sigma_2^2} - 1\right)^{\frac{1}{2}}} \leq 0$$

$$\rho_+ = \frac{e^{\sigma_1 \sigma_2} - 1}{\left(e^{\sigma_1^2} - 1\right)^{\frac{1}{2}} \left(e^{\sigma_2^2} - 1\right)^{\frac{1}{2}}} \geq 0$$

ρ_- and ρ_+ are not necessarily equal to -1 and 1. Example with $\sigma_1 = 1$ and $\sigma_2 = 3$:

Copula	$\rho(X_1, X_2)$	$\tau(X_1, X_2)$	$\varrho(X_1, X_2)$
$\mathbf{C}^-$	-0.008	-1	-1
$\rho = -0.7$	$\simeq 0$	-0.49	-0.68
$\mathbf{C}^\perp$	0	0	0
$\rho = 0.7$	$\simeq 0.10$	0.49	0.68
$\mathbf{C}^+$	0.16	1	1



Permissible range of $\rho(X_1, X_2)$
when X_1 and X_2 are two LN random variables

- Using an idea of Ferguson [1994], Nelsen [1999] defines the following copula

$$\mathbf{C}(u_1, u_2) = \begin{cases} u_1 & 0 \le u_1 \le \frac{1}{2}u_2 \le \frac{1}{2} \\ \frac{1}{2}u_2 & 0 \le \frac{1}{2}u_2 \le u_1 \le 1 - \frac{1}{2}u_2 \\ u_1 + u_2 - 1 & \frac{1}{2} \le 1 - \frac{1}{2}u_2 \le u_1 \le 1 \end{cases}$$

We have $\operatorname{cov}(U_1, U_2) = 0$, but $\Pr\{U_2 = 1 - |2U_1 - 1|\} = 1$, i.e. "*the two random variables can be uncorrelated although one can be predicted perfectly from the other*".

3 Understanding the dependence: three examples

- Mispecifications of marginals and dependence
- Dependence in multi-assets options
- Temporal dependence in Markov processes

3.1 Mispecifications of marginals and dependence

Example of Costinot, Roncalli and Teïletche [2000]

- Normal copula + Gaussian marginals = Gaussian distribution.

$$\hat{\rho} = \begin{bmatrix} 1 & 0.158 & 0.175 \\ & 1 & 0.0589 \\ & & 1 \end{bmatrix}$$

It means that

$$\mathbf{C}_{(\text{CAC40,NIKKEI})} \succ \mathbf{C}_{(\text{CAC40,DowJones})} \succ \mathbf{C}_{(\text{NIKKEI,DowJones})}$$

- Normal copula + Empirical marginals.

$$\hat{\rho} = \begin{bmatrix} 1 & 0.207 & 0.157 \\ & 1 & 0.0962 \\ & & 1 \end{bmatrix}$$

In this case, we have

$$\mathbf{C}_{(\text{CAC40,DowJones})} \succ \mathbf{C}_{(\text{CAC40,NIKKEI})} \succ \mathbf{C}_{(\text{NIKKEI,DowJones})}$$

3.2 Dependence in multi-assets options

Ref.: Bikos [2000], Cherubini and Luciano [2000], Durrleman [2001], Rosenberg [2000].

Vanilla options contain information on the future distribution of $S(T)$. This information (RND) is actually used in monetary policy (see BIS [1999]) (option prices = "forward-looking" indicators).

Bikos [2000]: options on multi-assets contain information on the future distribution of $\mathbf{S}(T) = \begin{pmatrix} S_1(T) & \cdots & S_N(T) \end{pmatrix}^\top$.

Let $\mathbb{Q}_n$ and $\mathbb{Q}$ be the risk-neutral probability distributions of $S_n(T)$ and $\mathbf{S}(T)$. With arbitrage theory, we can show that

$$\mathbb{Q}(+\infty, \ldots, +\infty, S_n(T), +\infty, \ldots, +\infty) = \mathbb{Q}_n(S_n(T))$$

$\Rightarrow$ The margins of $\mathbb{Q}$ are the RND $\mathbb{Q}_n$ of Vanilla options.

How to build ‘forward-looking” indicators for the dependence ?

1. estimate the univariate RND $\widehat{\mathbb{Q}}_n$ using Vanilla options;
2. estimate the copula $\hat{\mathbf{C}}$ using multi-assets options by imposing that $\mathbb{Q}_n = \widehat{\mathbb{Q}}_n$;
3. derive "forward-looking" indicators directly from $\hat{\mathbf{C}}$.

Breeden et Litzenberger [1978] remark that European option prices permit to caracterize the probability distribution of $S_n(T)$

$$\begin{aligned} \phi(T,K) &:= 1 + e^{r(T-t_0)} \frac{\partial C(T,K)}{\partial K} \\ &= \Pr\{S_n(T) \leq K\} \\ &= \mathbb{Q}_n(K) \end{aligned}$$

Durrleman [2000] extends this result in the bivariate case:

1. for a call max option, $\phi(T,K)$ is the diagonal section of the copula

$$\phi(T,K) = \mathbf{C}\left(\mathbb{Q}_1(K), \mathbb{Q}_2(K)\right)$$

2. for a spread option, we have

$$\phi(T,K) = \int_0^{+\infty} \partial_1 \mathbf{C}\left(\mathbb{Q}_1(x), \mathbb{Q}_2(x+K)\right) \, \mathrm{d}\mathbb{Q}_1(x)$$

⇒ Other results are derived in Durrleman [2001] (bounds, general kernel pricing, etc.)

Computation of the implied parameter $\widehat{\rho}$ in a spread option:

- BS model: LN distribution calibrated with ATM options; Kernel pricing = LN distributions + Normal copula

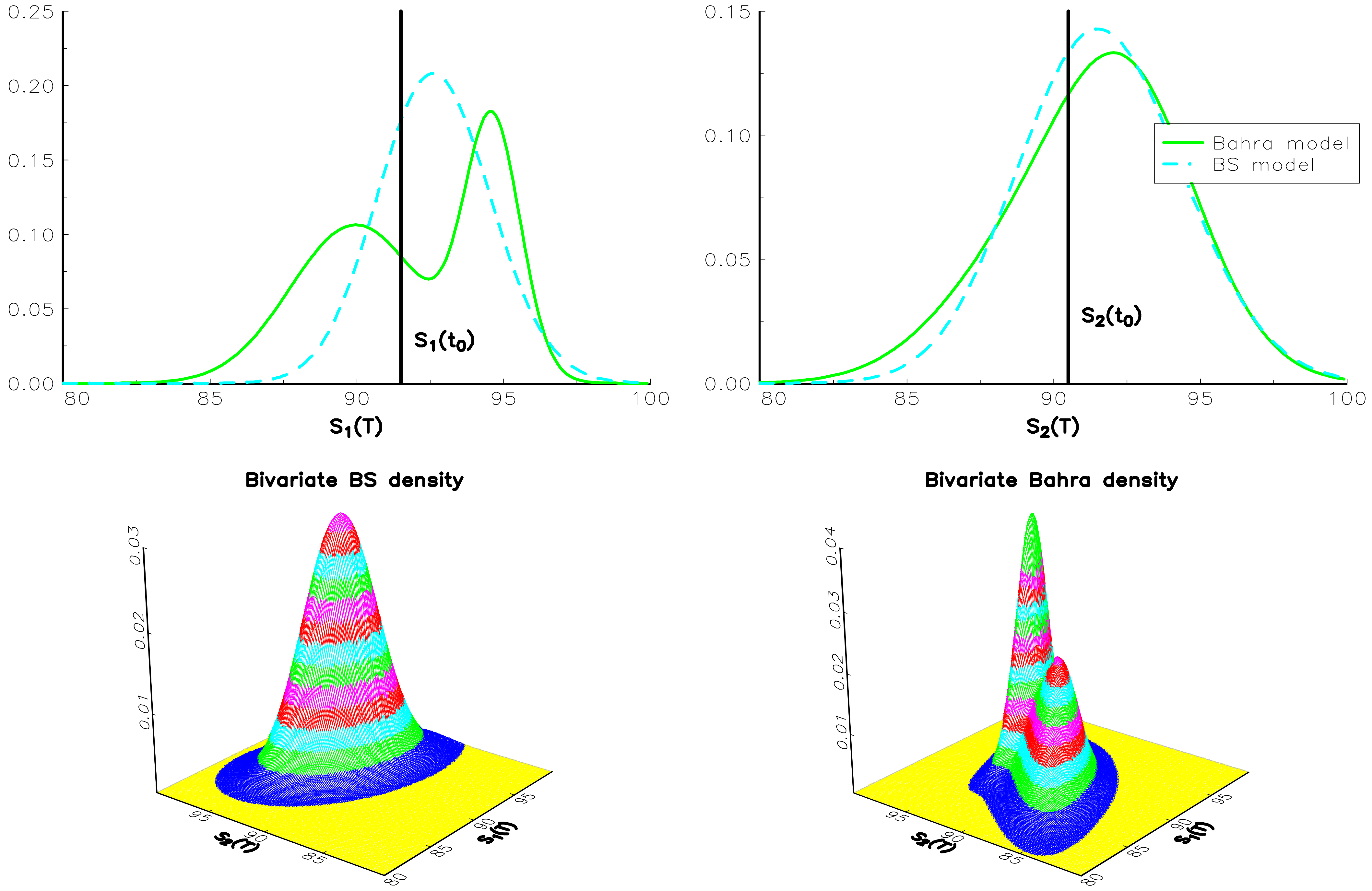
$$\widehat{\rho}_1 = -0.341$$

- Bahra model: mixture of LN distributions calibrated with eight European prices; Kernel pricing = MLN distributions + Normal copula

$$\widehat{\rho}_2 = 0.767$$

Remark 1 *$\widehat{\rho}_1$ and $\widehat{\rho}_2$ are parameters of the Normal Copula. $\widehat{\rho}_1$ is a Pearson correlation, not $\widehat{\rho}_2$.*

$\Rightarrow$ BS model: negative dependence / Bahra model: positive dependence.



A spread option example

3.3 Temporal dependence in Markov processes

- Markov operators and 2-copulas
- Markov operators and stochastic kernels
- Markov processes and $*$ product of 2-copulas
  - The Brownian copula
  - Understanding the temporal dependence structure of diffusion processes
  - New interpretation of properties of Markov processes

3.3.1 Markov operators and 2-copulas

Definition 2 (Brown [1966, p. 15]) *Let* $(\Omega, \mathcal{F}, P)$ *be a probabilistic space. A linear operator* $\mathbf{T} : L^{\infty}(\Omega) \to L^{\infty}(\Omega)$ *is a Markov operator if*

(a) $\mathbf{T}$ *is positive i.e.* $\mathbf{T}[f] \geq 0$ *whenever* $f \geq 0$*;*

(b) 1 *is a fixed point of* $\mathbf{T}$*.*

(c) $\mathbb{E}[\mathbf{T}[f]] = \mathbb{E}[f]$ *for every* $f \in L^{\infty}(\Omega)$*.*

The relationship between Markov operator $\mathbf{T}$ and 2-copula functions $\mathbf{C}$ is given by the following two lemmas:

Lemma 1 (Olsen, Darsow and Nguyen [1996, lemma 2.1, p. 249]) *For a copula $\mathbf{C}$, the operator defined by*

$$\mathbf{T}[f](x) = \frac{\mathrm{d}}{\mathrm{d}x} \int_0^1 \partial_2 \mathbf{C}(x, y) f(y) \, \mathrm{d}y$$

is a Markov operator on $L^\infty([0,1])$.

Lemma 2 (Olsen, Darsow and Nguyen [1996, lemma 2.2, p. 251]) *Let $\mathbf{T}$ be a Markov operator on $L^\infty([0,1])$. The two place function defined by*

$$\mathbf{C}(x, y) = \int_0^x \mathbf{T}\left[\mathbf{1}_{[0,y]}\right](s) \, \mathrm{d}s$$

is a 2-copula.

The multiplication product of copulas have been defined by Darsow, Nguyen and Olsen [1992] in the following manner

$$\begin{array}{rcl} \mathbf{I}^2 & \longrightarrow & \mathbf{I} \\ (x,y) & \longmapsto & (\mathbf{C}_1 * \mathbf{C}_2)(x,y) = \int_0^1 \partial_2 \mathbf{C}_1(x,s)\, \partial_1 \mathbf{C}_2(s,y)\, \mathrm{d}s \end{array}$$

The transposition of copula corresponds to the mapping function $\mathbf{C}^\top(x,y) = \mathbf{C}(y,x)$.

The adjoint $\mathbf{T}^\dagger$ of the Markov operator $\mathbf{T}$ is the operator such that we verify that (Brown [1966])

$$\int_{[0,1]} f_1(x)\, \mathbf{T}[f_2](x)\, m(\mathrm{d}x) = \int_{[0,1]} f_2(x)\, \mathbf{T}^\dagger[f_1](x)\, m(\mathrm{d}x)$$

In terms of copulas, we have

$$\mathbf{T}^\dagger[f](x) = \frac{\mathrm{d}}{\mathrm{d}y} \int_0^1 \partial_1 \mathbf{C}(x,y)\, f(x)\, \mathrm{d}x$$

3.3.2 Markov operators and stochastic kernels

Using the Kantorovitch-Vulich-Ryff representation, Foguel [1969] shows that $\mathbf{T}\langle\mathbf{K}_1\rangle \circ \mathbf{T}\langle\mathbf{K}_2\rangle$ is a Markov operator with kernel $\mathbf{K} = \mathbf{K}_1 * \mathbf{K}_2$ defined by

$$\mathbf{K}(x,y) := (\mathbf{K}_1 * \mathbf{K}_2)(x,y) = \int_{[0,1]} \mathbf{K}_1(x,s)\,\mathbf{K}_2(s,y)\,\mathrm{d}s$$

Let P be the stochastic transition function of a Markov process, we can deduce that

$$\mathbf{T}\left\langle P_{s,t}\right\rangle = \mathbf{T}\left\langle P_{s,\theta} * P_{\theta,t}\right\rangle = \mathbf{T}\left\langle P_{s,\theta}\right\rangle \circ \mathbf{T}\left\langle P_{\theta,t}\right\rangle$$

with

$$\begin{aligned} P_{s,t}(x,\mathcal{A}) := \left(P_{s,\theta} * P_{\theta,t}\right)(x,\mathcal{A}) &= \int_{\Omega} P_{s,\theta}(x,\mathrm{d}y)\,P_{\theta,t}(y,\mathcal{A}) \\ &= P^{\star}(x,\mathcal{A}) \end{aligned}$$

$\Rightarrow$ this is the **Chapman-Kolmogorov equation**.

3.3.3 Markov processes and $*$ product of 2-copulas

Darsow, Nguyen and Olsen [1992] prove the following theorem:

Theorem 3 *Let $X = \{X_t, \mathcal{F}_t; t \geq 0\}$ be a stochastic process and let $\mathbf{C}_{s,t}$ denote the copula of the random variables X_s and X_t. Then the following are equivalent*

(i) The transition probabilities $P_{s,t}(x, \mathcal{A}) = \Pr\{X_t \in \mathcal{A} \mid X_s = x\}$ satisfy the Chapman-Kolmogorov equations

$$P_{s,t}(x, \mathcal{A}) = \int_{-\infty}^{\infty} P_{s,\theta}(x, \mathrm{d}y)\, P_{\theta,t}(y, \mathcal{A})$$

for all $s < \theta < t$ and almost all $x \in \mathbb{R}$.

(ii) For all $s < \theta < t$,

$$\mathbf{C}_{s,t} = \mathbf{C}_{s,\theta} * \mathbf{C}_{\theta,t} \qquad (1)$$

In the conventional approach, one specifies a Markov process by giving the initial distribution μ and a family of transition probabilities $P_{s,t}(x, \mathcal{A})$ satisfying the Chapman-Kolmogorov equations. In our approach, one specifies a Markov process by giving all of the marginal distributions and a family of 2-copulas satisfying (1). Ours is accordingly an alternative approach to the study of Markov processes which is different in principle from the conventional one. Holding the transition probabilities of a Markov process fixed and varying the initial distribution necessarily varies all of the marginal distributions, but holding the copulas of the process fixed and varying the initial distribution does not affect any other marginal distribution (Darsow, Nguyen and Olsen [1992]).

The Brownian copula

$$\mathbf{C}_{s,t}(u_1, u_2) = \int_0^{u_1} \Phi\left(\frac{\sqrt{t}\Phi^{-1}(u_2) - \sqrt{s}\Phi^{-1}(u)}{\sqrt{t-s}}\right) \mathrm{d}u$$

Understanding the dependence structure of diffusion processes

The copula of a Geometric Brownian motion is the Brownian copula.

The Ornstein-Uhlenbeck copula is

$$\mathbf{C}_{s,t}(u_1, u_2) = \int_0^{u_1} \Phi\left(\frac{\hbar(t_0, s, t)\, \Phi^{-1}(u_2) - \hbar(t_0, s, s)\, \Phi^{-1}(u)}{\hbar(s, s, t)}\right) \mathrm{d}u$$

with

$$\hbar(t_0, s, t) = \sqrt{e^{2a(t-s)} - e^{-2a(s-t_0)}}$$

Remark 2 *A new interpretation of the parameter a follows. For physicists, a is the mean-reverting coefficient. From a copula point of view, this parameter measures the dependence between the random variables of the diffusion process. The bigger this parameter, the less dependent the random variables.*

New interpretation of properties of Markov processes

Theorem 4 (Darsow, Nguyen and Olsen [1992, theorem 5.1, p. 622]) *The set $\mathcal{C}$ is a symmetric Markov algebra under $*$ and $^\top$ as previously defined. The unit and null elements are $\mathbf{C}^\perp$ and $\mathbf{C}^+$.*

- C_{st} is a left invertible copula $\Leftrightarrow$ the markov process is deterministic.
- C_{st} is idempotent $\Leftrightarrow$ the Markov process is conditionally independent.

4 An open field for risk management

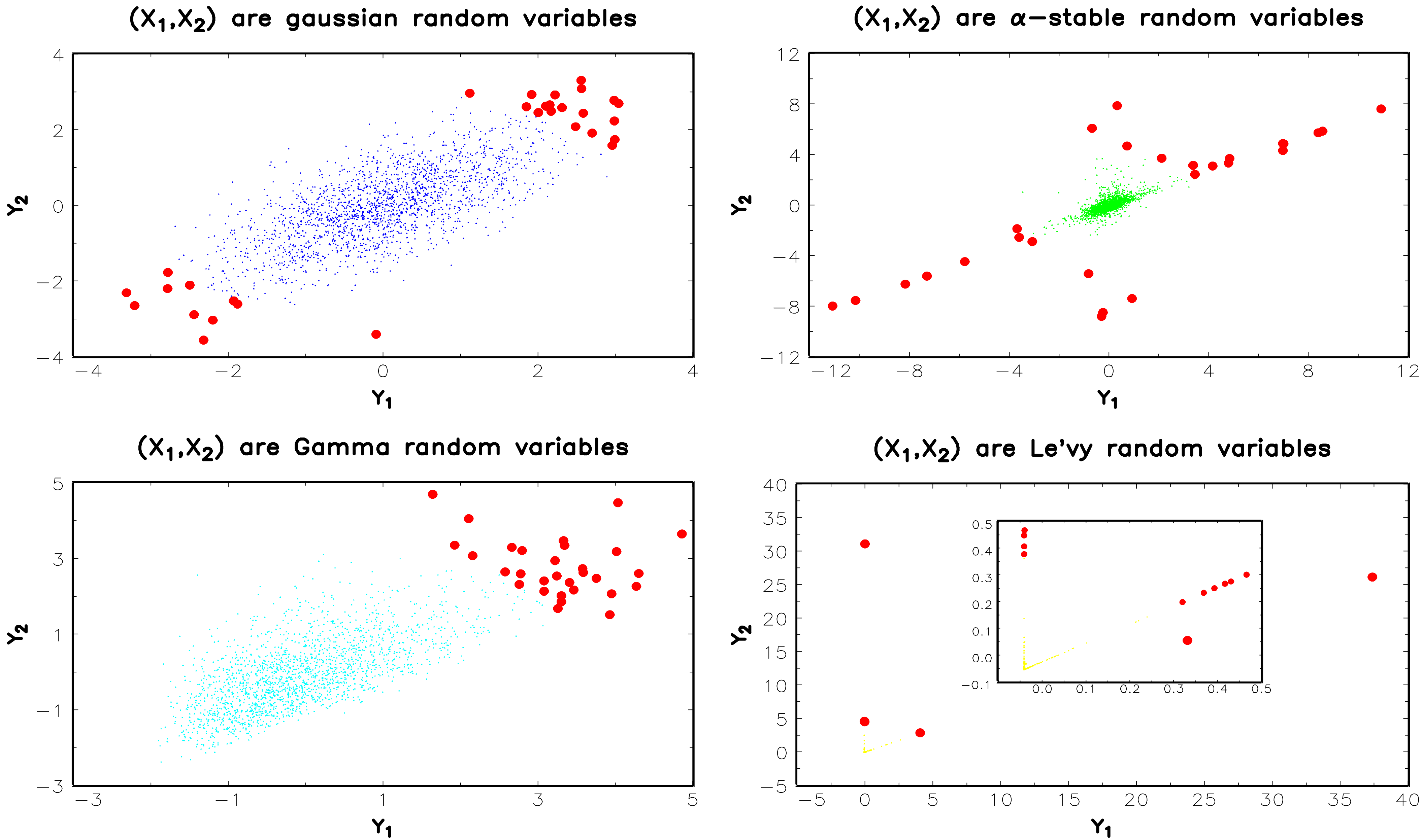
- Economic capital adequacy
- Market risk
- Credit risk
- Operational risk

$\Rightarrow$ see BDNRR [2000] for a more detailed presentation (in particular for stress-testing, multivariate extreme value theory and operational risk) and DNRa [2000] for the problem of quantile aggregation.

4.1 Economic capital adequacy

With copulas, it appears that the risk can be splitted into two parts: the individual risks and the dependence structure between them.

- **Coherent multivariate statistical model = Coherent model for individual risks + coherent dependence function**
- **Coherent model for individual risks** = taking into account fat-tailed distributions, etc.
- **coherent dependence function** = understanding the aggregation of quantiles of the individual risks.



Bivariate distributions with same first and second moments
= Gaussian VaRs are equal

⇒ The influence of margins

Rating	*VaR*	BBB	A	AA	AAA
α	99%	99.75%	99.9%	99.95%	99.97%
Return time	100 days	400 days	4 years	8 years	13 years
$\frac{\Phi^{-1}(\alpha)}{\Phi^{-1}(0.99)}$	1	1.20	1.33	1.41	1.48
$\frac{t_4^{-1}(\alpha)}{t_4^{-1}(0.99)}$	1	1.49	1.91	2.30	2.62

⇒ The influence of the dependence function: If a bivariate copula $\mathbf{C}$ is such that*

$$\lim_{u \to 1} \frac{\bar{\mathbf{C}}(u, u)}{1 - u} = \lambda$$

exists, then $\mathbf{C}$ has **upper tail dependence** for $\lambda \in (0, 1]$ and no upper tail dependence for $\lambda = 0$.

*$\bar{\mathbf{C}}$ is the joint survival function, that is

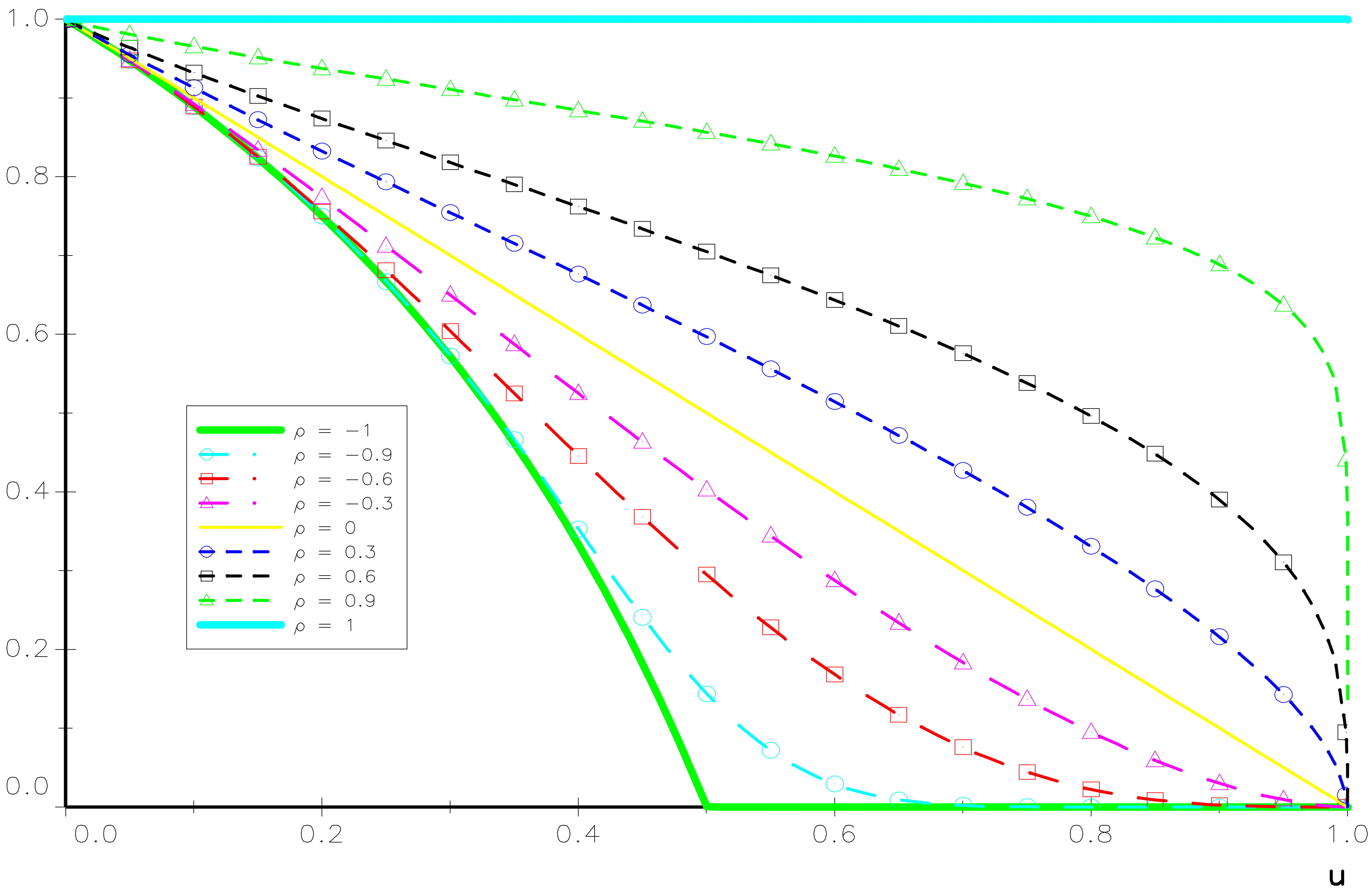
$$\bar{\mathbf{C}}(u_1, u_2) = 1 - u_1 - u_2 + \mathbf{C}(u_1, u_2)$$

Remark 3 *The measure λ is the probability that one variable is extreme given that the other is extreme.*

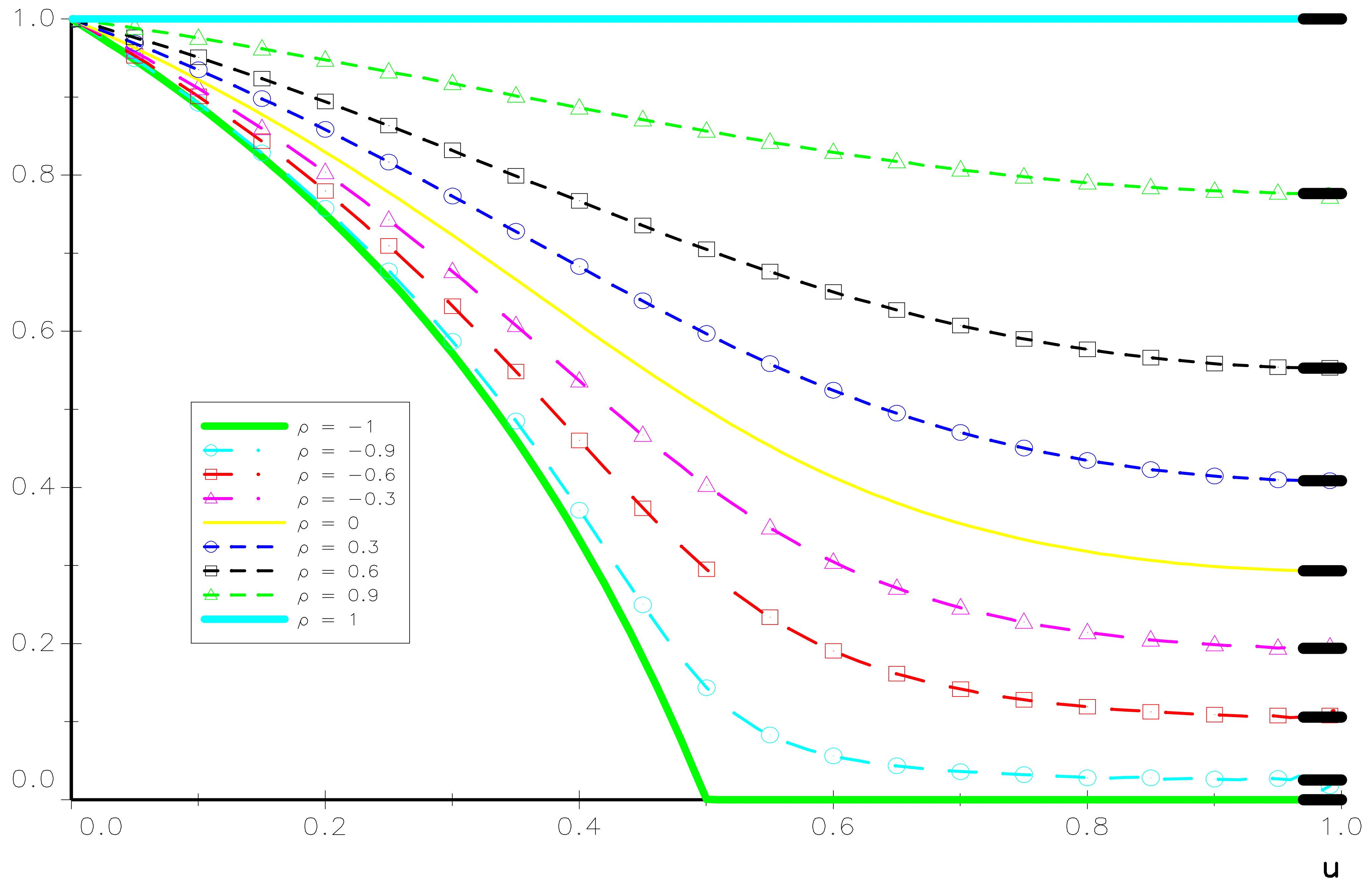
$\Rightarrow$ Coles, Currie and Tawn [1999] define the quantile-dependent measure of dependence as follows

$$\lambda(u) = \Pr\{U_1 > u | U_2 > u\} = \frac{\bar{\mathbf{C}}(u,u)}{1-u}$$

1. Normal copula $\Rightarrow$ extremes are asymptotically independent for $\rho \neq 1$, i.e $\lambda = 0$ for $\rho < 1$.
2. Student copula $\Rightarrow$ extremes are asymptotically dependent for $\rho \neq -1$.



Quantile−dependent measure λ(u) for the Normal copula



λ(u) for the Student copula and ν = 1

4.2 Market risk

Copulas = a powerful tool for market risk measurement.

Copulas have been already incorporated in some software solutions:

- SAS Risk Dimensions
- Palisade @Risk

LME example:

	AL	AL-15	CU	NI	PB
P_1	1	1	1	1	1
P_2	-1	-1	-1	1	1
P_3	2	1	-3	4	5

- Gaussian margins and Normal copula

	90%	95%	99%	99.5%	99.9%
P_1	7.26	9.33	13.14	14.55	17.45
P_2	4.04	5.17	7.32	8.09	9.81
P_3	13.90	17.82	25.14	27.83	33.43

- Student margins ($\nu = 4$) and Normal copula

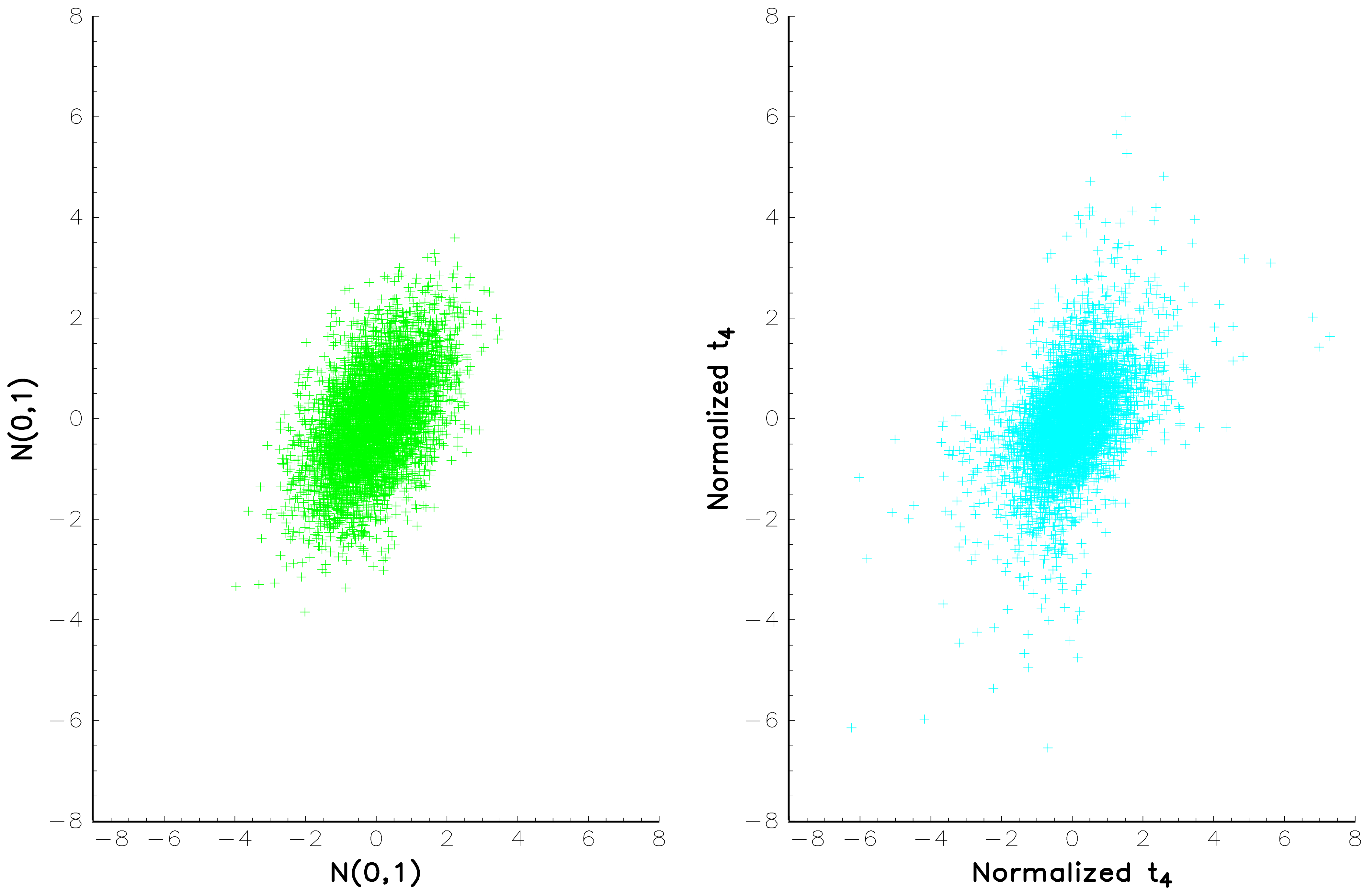
	90%	95%	99%	99.5%	99.9%
P_1	9.20	12.48	20.16	23.95	34.07
P_2	5.33	7.08	11.16	13.17	19.17
P_3	18.04	24.11	38.90	46.45	69.51

• Gaussian margins and Student copula ($\nu = 1$)

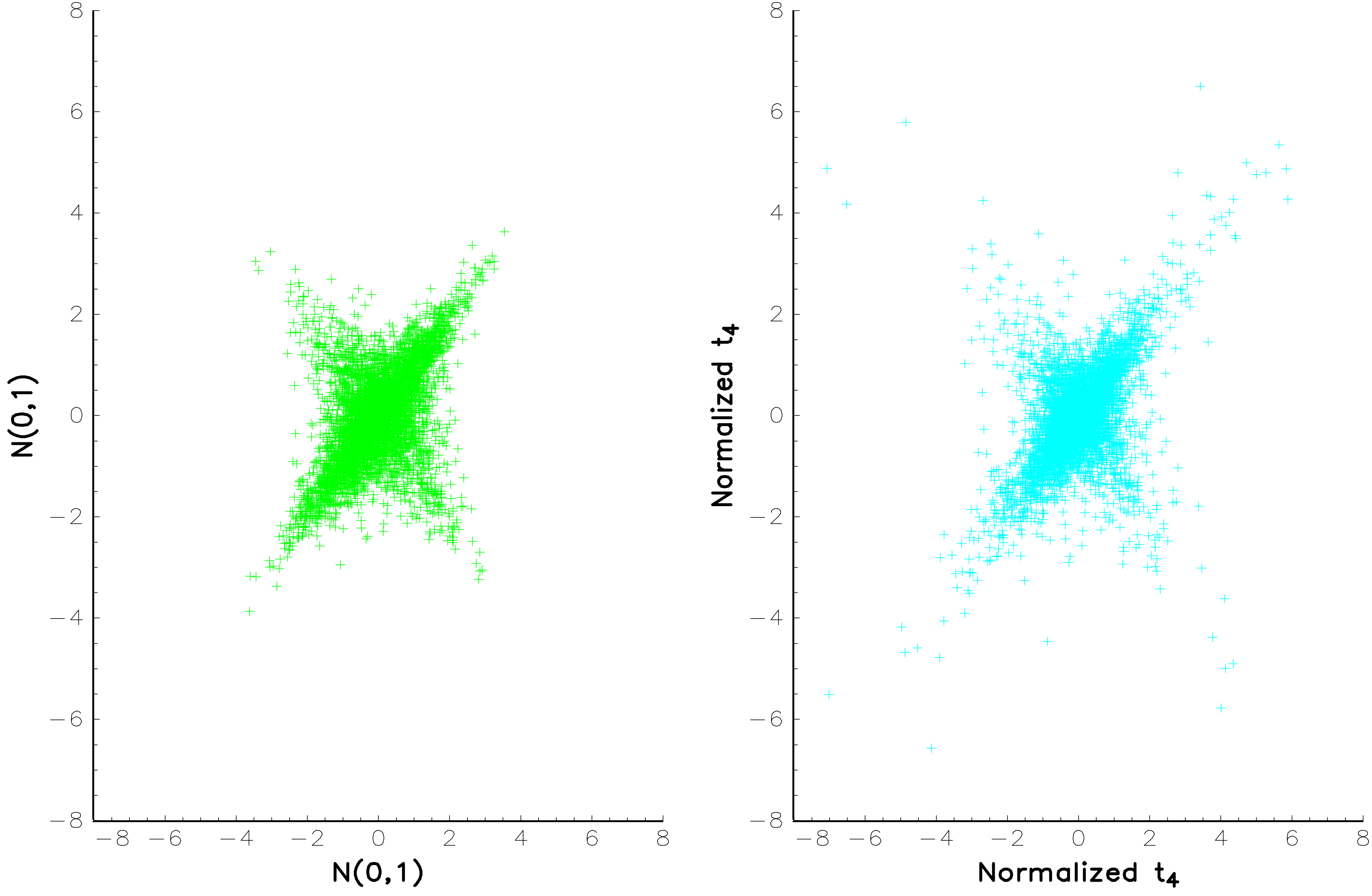
	90%	95%	99%	99.5%	99.9%
P_1	6.49	8.94	14.48	16.67	21.11
P_2	3.45	5.08	9.17	11.03	15.77
P_3	11.99	17.53	31.88	37.94	51.66

Value-at-risk based on Student margins and a Normal copula (Gauss software, Pentium III 550 Mhz, 100000 simulations)

Number of assets	Computational time
2	0.1 sc
10	24.5 sc
100	4 mn 7 sc
500	33 mn 22 sc
1000	1 hr 44 mn 45 sc



5000 simulations with a Normal copula (ρ = 0.5)



5000 simulations with a Student copula ($\rho = 0.5$, $\nu = 1$)

5 Modelling credit risk

Some articles are available, see for example Coutant, Martineu, Messines, Riboulet and Roncalli [2001], Li [2000], Lindskog [2000], Lindskog and McNeil [2000] and Nyfeler [2000].

Other articles are still in progress ⇒ **credit risk = one of the most important topic for financial applications of copulas**.

- Dependence in credit risk models
  - The credit migration approach (CreditMetrics)
  - The actuarial approach (CreditRisk+)
  - The survival approach
- Pricing of credit derivatives

5.1 Dependence in credit risk models

Portfolio with liquids credits $\neq$ Portfolio with not liquids credits.

$\Rightarrow$ downgrading risk $\neq$ default risk.

What is the influence of introducing a dependence function?

1. impact on the joint migration probability distribution;
2. impact on the joint survival distribution.

5.1.1 The credit migration approach (CreditMetrics)

Notations N = number of credits in the portfolio.
$\left(\pi_{i,j}\right)$ = rating transition matrix from one state to another.
S = set of the eight states $\{\text{AAA}, \text{AA}, \text{A}, \text{BBB}, \text{BB}, \text{B}, \text{CCC}, \text{D}\}$.
$S^{\star} = \{1, \ldots, 8\}$ with $1 \to$ D, $2 \to$ CCC, $3 \to$ B, etc. ($8 \to$ AAA).
π_i = distribution of the initial rating i (R_i is the corresponding random variable).
π_i = discrete probability distribution defined by $\{j, \pi_i(j)\}$ with $\pi_i(j) = \Pr\{R_i \leq j\} = \sum_{k=1}^{j} \pi_{i,k}$ and $j \in S^{\star}$.
$\mathbf{P}(i_1, \ldots, i_N; j_1, \ldots, j_N)$ = joint migration probability distribution

$$\mathbf{P}(i_1, \ldots, i_N; j_1, \ldots, j_N) = \Pr\left\{R_{i_1} \leq j_1, \ldots, R_{i_N} \leq j_N\right\}$$

$p(i_1, \ldots, i_N; j_1, \ldots, j_N)$ = joint migration probability mass function

$$p(i_1, \ldots, i_N; j_1, \ldots, j_N) = \Pr\left\{R_{i_1} = j_1, \ldots, R_{i_N} = j_N\right\}$$

The bivariate migration probability distribution

CreditMetrics uses a gaussian random variable Z_i

$$\pi_i(j) = \Pr\left\{Z_i \leq z_j^{(i)}\right\}$$

Gupton, Finger and Bhatia [1997] define then the joint migration probability distribution by

$$\mathbf{P}(i_1, i_2; j_1, j_2) = \Pr\left\{Z_{i_1} \leq z_{j_1}^{(i_1)}, Z_{i_2} \leq z_{j_2}^{(i_2)}\right\} = \Phi\left(z_{j_1}^{(i_1)}, z_{j_2}^{(i_2)}; \rho\right)$$

where $\rho = \rho(C_1, C_2)$ is the 'asset return correlation' of C_1 and C_2.

Remark that $\pi_i(R_i) = \Phi(Z_i)$. It comes that

$$\mathbf{P}(i_1, i_2; j_1, j_2) = \Phi\left(\Phi^{-1}\left(\pi_{i_1}(j_1)\right), \Phi^{-1}\left(\pi_{i_2}(j_2)\right); \rho\right)$$

We can then write $\mathbf{P}(i_1, i_2; j_1, j_2)$ as a function of the bivariate Normal copula $\mathbf{C}$

$$\mathbf{P}(i_1, i_2; j_1, j_2) = \mathbf{C}\left(\pi_{i_1}(j_1), \pi_{i_2}(j_2); \rho\right)$$

The multivariate migration probability distribution

Let $\mathbf{C}$ be a copula function. We have

$$\mathbf{P}\left(i_1, \ldots, i_N; j_1, \ldots, j_N\right) = \mathbf{C}\left(\pi_{i_1}\left(j_1\right), \ldots, \pi_{i_N}\left(j_N\right)\right)$$

The probability mass function $p\left(i_1, \ldots, i_N; j_1, \ldots, j_N\right)$ is given by the Radon-Nikodym density of the copula*

$$\sum_{k_1=0}^{1} \cdots \sum_{k_N=0}^{1} (-1)^{k_1+\cdots+k_N} \mathbf{C}\left(\pi_{i_1}\left(j_1 - k_1\right), \ldots, \pi_{i_N}\left(j_N - k_N\right)\right)$$

Because a copula is a grounded function, we deduce that the joint default probability is

$$p\left(i_1, \ldots, i_N; \mathsf{D}, \ldots, \mathsf{D}\right) = \mathbf{C}\left(\pi_{i_1}(1), \ldots, \pi_{i_N}(1)\right)$$

It comes that

$$\mathbf{C}^{-}\left(\pi_{i_1}(1), \ldots, \pi_{i_N}(1)\right) \leq p\left(i_1, \ldots, i_N; \mathsf{D}, \ldots, \mathsf{D}\right) \leq \mathbf{C}^{+}\left(\pi_{i_1}(1), \ldots, \pi_{i_N}(1)\right)$$

*We use the convention $(-1)^0 = 1$.

Some illustrations

- migration probabilities $p(i_1, i_2; j_1, j_2)$
- joint default probability $p(i_1, i_2; \mathsf{D}, \mathsf{D})$
- discrete default correlation

$$\rho_{\mathsf{D}}(C_1, C_2) = \frac{p(i_1, i_2; \mathsf{D}, \mathsf{D}) - \pi_{i_1,1}\pi_{i_2,1}}{\sqrt{\pi_{i_1,1}\left(1 - \pi_{i_1,1}\right)\pi_{i_2,1}\left(1 - \pi_{i_2,1}\right)}}$$

Remark 4 The discrete default correlation is not a good measure of the dependence between defaults. *We note that*

$$-1 < \rho_{\mathsf{D}}^{-}(C_1, C_2) \leq \rho_{\mathsf{D}}(C_1, C_2) \leq \rho_{\mathsf{D}}^{+}(C_1, C_2) < 1$$

Even if the dependence is maximal ($\mathbf{C} = \mathbf{C}^{+}$), $\rho_{\mathsf{D}}(C_1, C_2)$ can be very small (less than 10%).

$\Rightarrow$ see CMMRR [2001] for the link between $\rho_{\mathsf{D}}(C_1, C_2)$ and a 2×2 contingency table.

	Final rating							
Initial rating	AAA	AA	A	BBB	BB	B	CCC	D
AAA	92.54	6.48	0.86	0.06	0.06	0.00	0.00	0.00
AA	0.63	91.87	6.64	0.65	0.06	0.11	0.04	0.00
A	0.08	2.26	91.66	5.11	0.61	0.23	0.01	0.04
BBB	0.05	0.27	5.84	87.74	4.74	0.98	0.16	0.22
BB	0.04	0.11	0.64	7.85	81.14	8.27	0.89	1.06
B	0.00	0.11	0.30	0.42	6.75	83.07	3.86	5.49
CCC	0.19	0.00	0.38	0.75	2.44	12.03	60.71	23.50
D	0	0	0	0	0	0	0	100

Table 1: S&P one-year transition matrix (in %)

	Initial rating						
Final rating	AAA	AA	A	BBB	BB	B	CCC
D	0.00	0.00	0.04	0.22	1.06	5.49	23.50
CCC	0.00	0.04	0.05	0.38	1.95	9.35	84.21
B	0.00	0.15	0.28	1.36	10.22	92.42	96.24
BB	0.06	0.21	0.89	6.10	91.36	99.17	98.68
BBB	0.12	0.86	6.00	93.84	99.21	99.59	99.43
A	0.98	7.50	97.66	99.68	99.85	99.89	99.81
AA	7.46	99.37	99.92	99.95	99.96	100.00	99.81
AAA	100.00	100.00	100.00	100.00	100.00	100.00	100.00

Table 2: S&P one-year marginal probability distributions (in %)

	Final rating of C_2								
Final rating of C_1	AAA	AA	A	BBB	BB	B	CCC	D	$\pi_{i_1,j}$
AAA	0.00	0.00	0.01	0.01	0.03	0.03	0.00	0.00	0.08
AA	0.00	0.03	0.07	0.08	0.67	1.40	0.00	0.00	2.26
A	0.00	0.07	0.22	0.34	6.02	77.79	3.20	4.01	91.66
BBB	0.00	0.00	0.00	0.00	0.02	3.42	0.54	1.13	5.11
BB	0.00	0.00	0.00	0.00	0.00	0.32	0.08	0.21	0.61
B	0.00	0.00	0.00	0.00	0.00	0.10	0.03	0.10	0.23
CCC	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.01
D	0.00	0.00	0.00	0.00	0.00	0.01	0.00	0.02	0.04
$\pi_{i_2,j}$	0.00	0.11	0.30	0.42	6.75	83.07	3.86	5.49	100.00

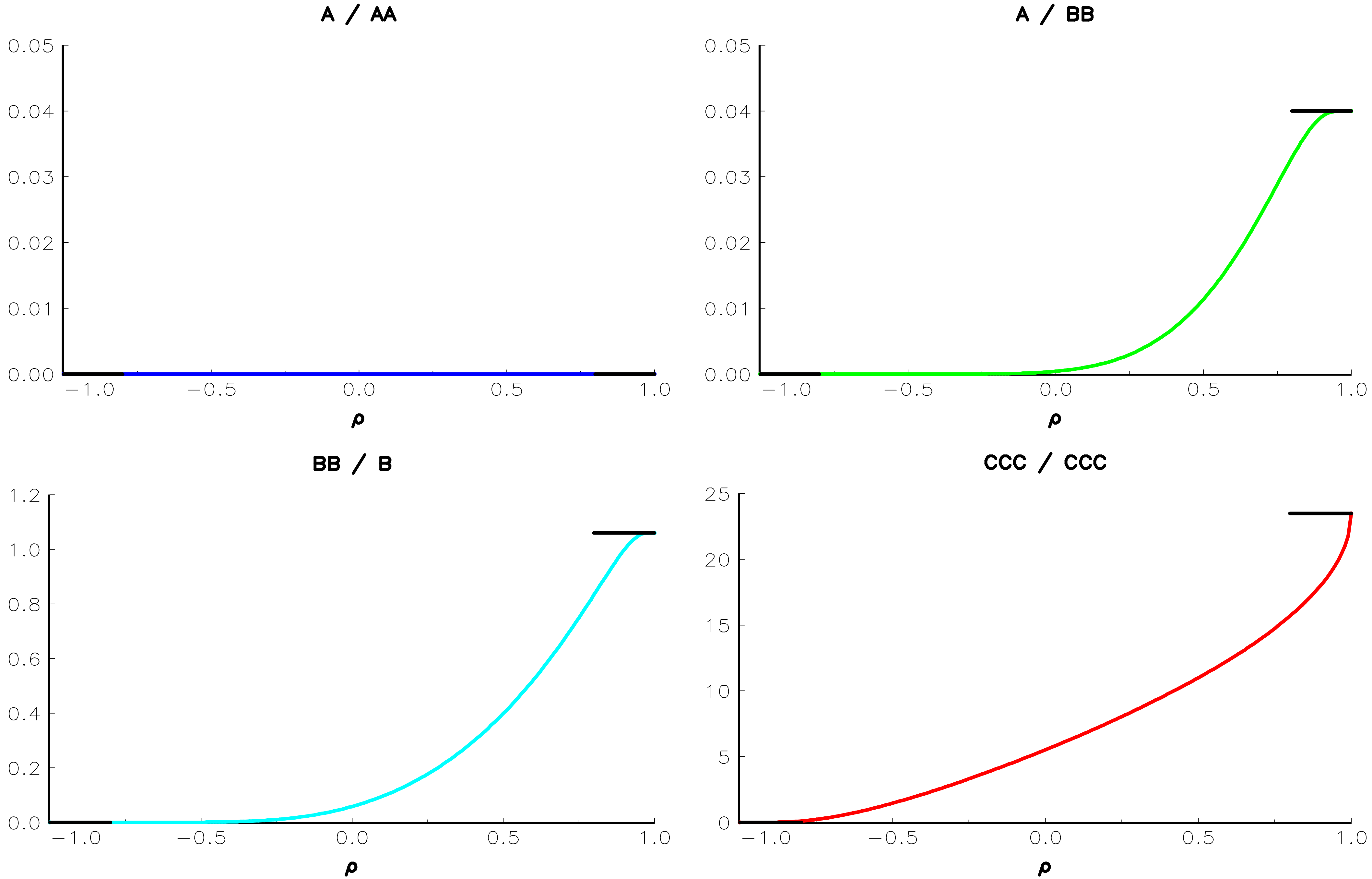
Table 3: Migration probabilities $p(i_1, i_2, j_1, j_2)$ (in %) with $i_1 = \mathsf{A}$, $i_2 = \mathsf{B}$ and $\rho(C_1, C_2) = 0.50$

	Final rating of C_2								
Final rating of C_1	AAA	AA	A	BBB	BB	B	CCC	D	$\pi_{i_1,j}$
AAA	0.00	0.00	0.00	0.00	0.00	0.03	0.01	0.04	0.08
AA	0.00	0.00	0.00	0.00	0.00	1.29	0.27	0.70	2.26
A	0.00	0.05	0.17	0.27	5.24	77.64	3.56	4.74	91.66
BBB	0.00	0.04	0.09	0.11	1.20	3.64	0.02	0.01	5.11
BB	0.00	0.01	0.02	0.02	0.20	0.35	0.00	0.00	0.61
B	0.00	0.01	0.01	0.01	0.09	0.11	0.00	0.00	0.23
CCC	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.01
D	0.00	0.00	0.00	0.00	0.02	0.01	0.00	0.00	0.04
$\pi_{i_2,j}$	0.00	0.11	0.30	0.42	6.75	83.07	3.86	5.49	100.00

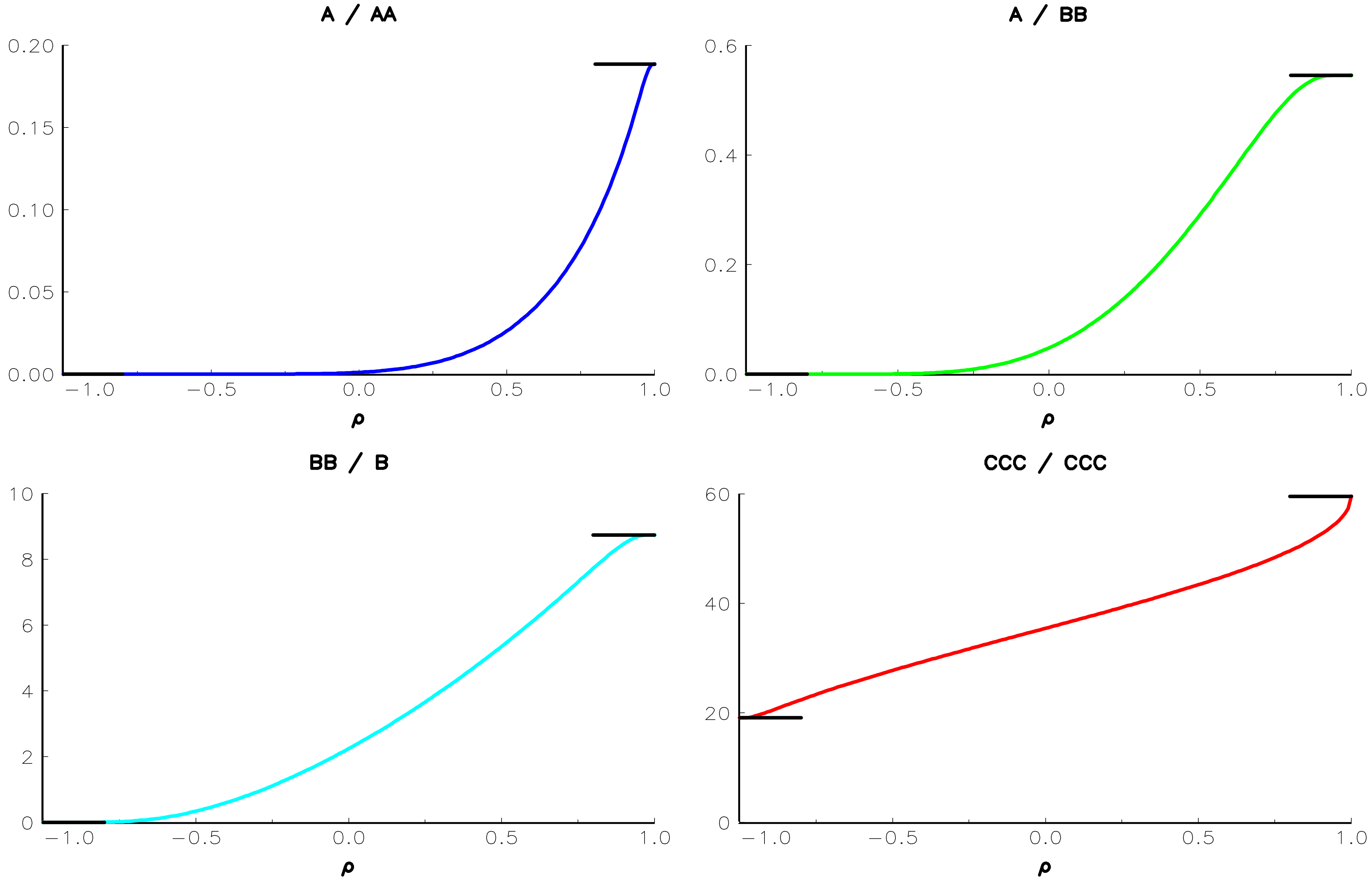
Table 4: Migration probabilities $p(i_1, i_2, j_1, j_2)$ (in %) with $i_1 = \mathsf{A}$, $i_2 = \mathsf{B}$ and $\rho(C_1, C_2) = -0.50$

	Final rating of C_2								
Final rating of C_1	AAA	AA	A	BBB	BB	B	CCC	D	$\pi_{i_1,j}$
AAA	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
AA	0.01	0.00	0.01	0.01	0.02	0.04	0.02	0.00	0.11
A	0.01	0.00	0.02	0.02	0.05	0.10	0.09	0.00	0.30
BBB	0.01	0.00	0.02	0.03	0.06	0.14	0.16	0.00	0.42
BB	0.07	0.00	0.13	0.23	0.61	1.98	3.50	0.22	6.75
B	0.08	0.00	0.21	0.46	1.69	9.63	53.11	17.89	83.07
CCC	0.00	0.00	0.00	0.00	0.01	0.08	1.84	1.93	3.86
D	0.00	0.00	0.00	0.00	0.00	0.06	1.98	3.44	5.49
$\pi_{i_2,j}$	0.19	0.00	0.38	0.75	2.44	12.03	60.71	23.50	100.00

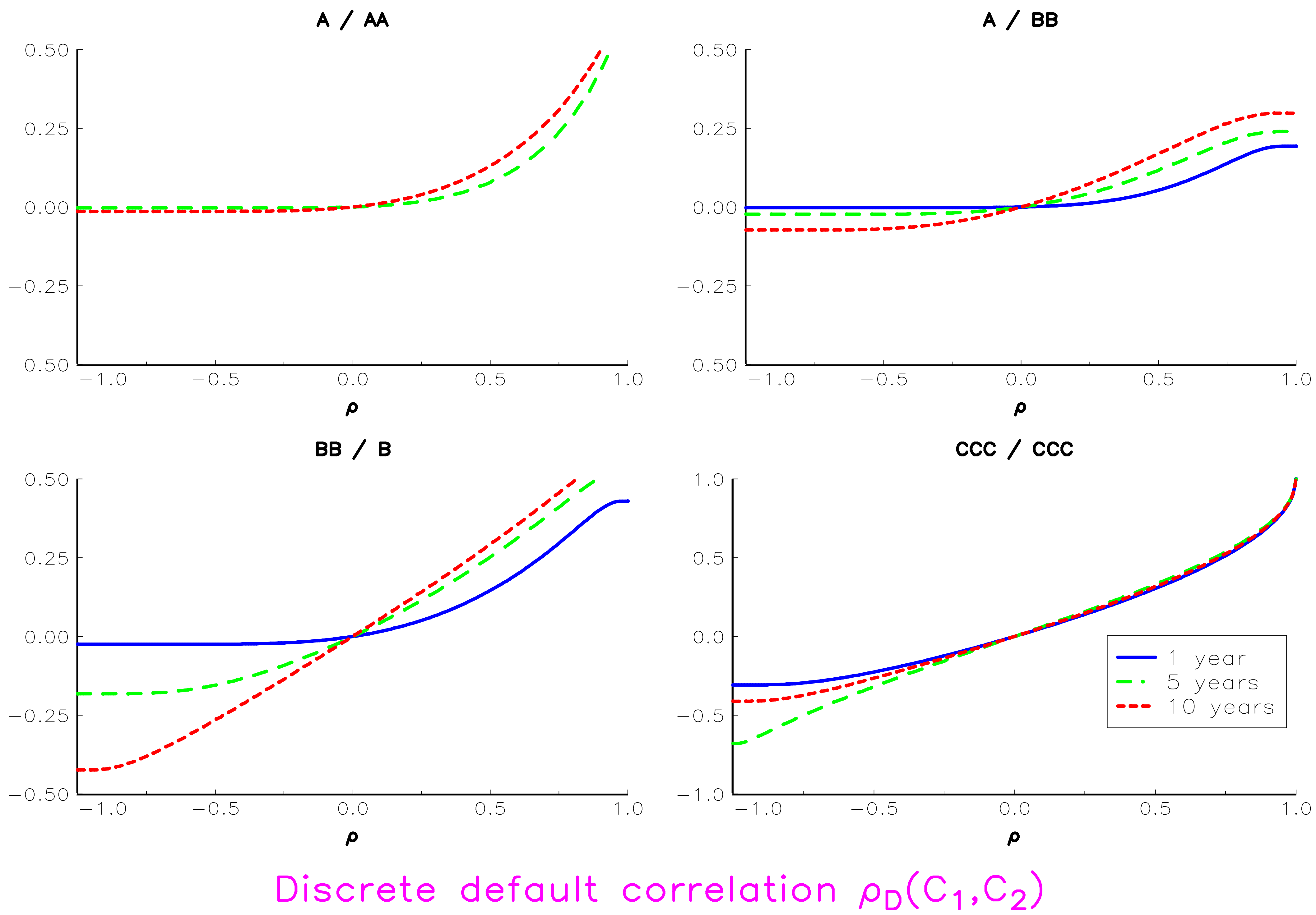
Table 5: Migration probabilities $p(i_1, i_2, j_1, j_2)$ (in %) with $i_1 = \mathsf{B}$, $i_2 = \mathsf{CCC}$ and $\rho(C_1, C_2) = 0.50$

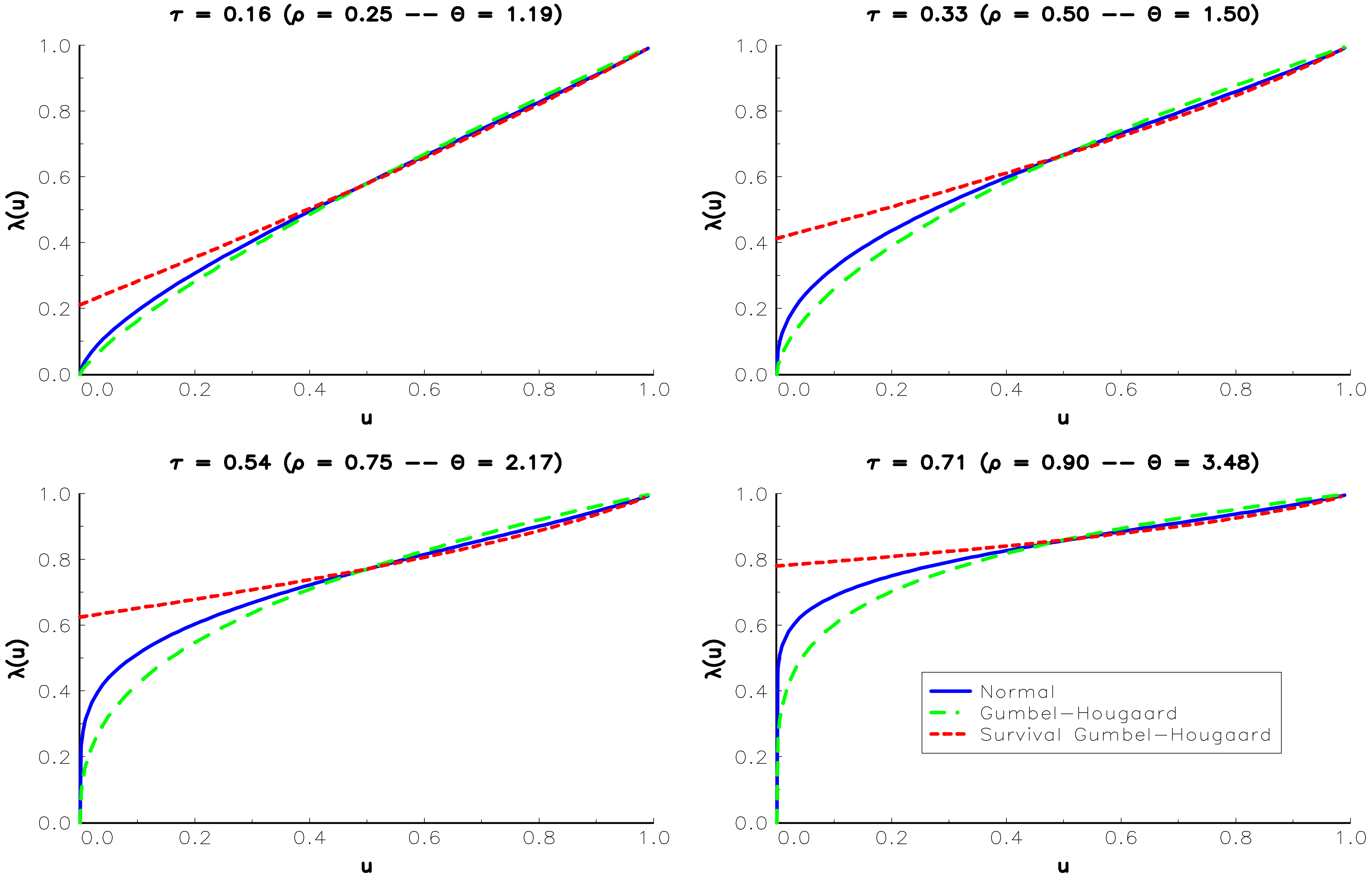


One−year joint default probability (in %)

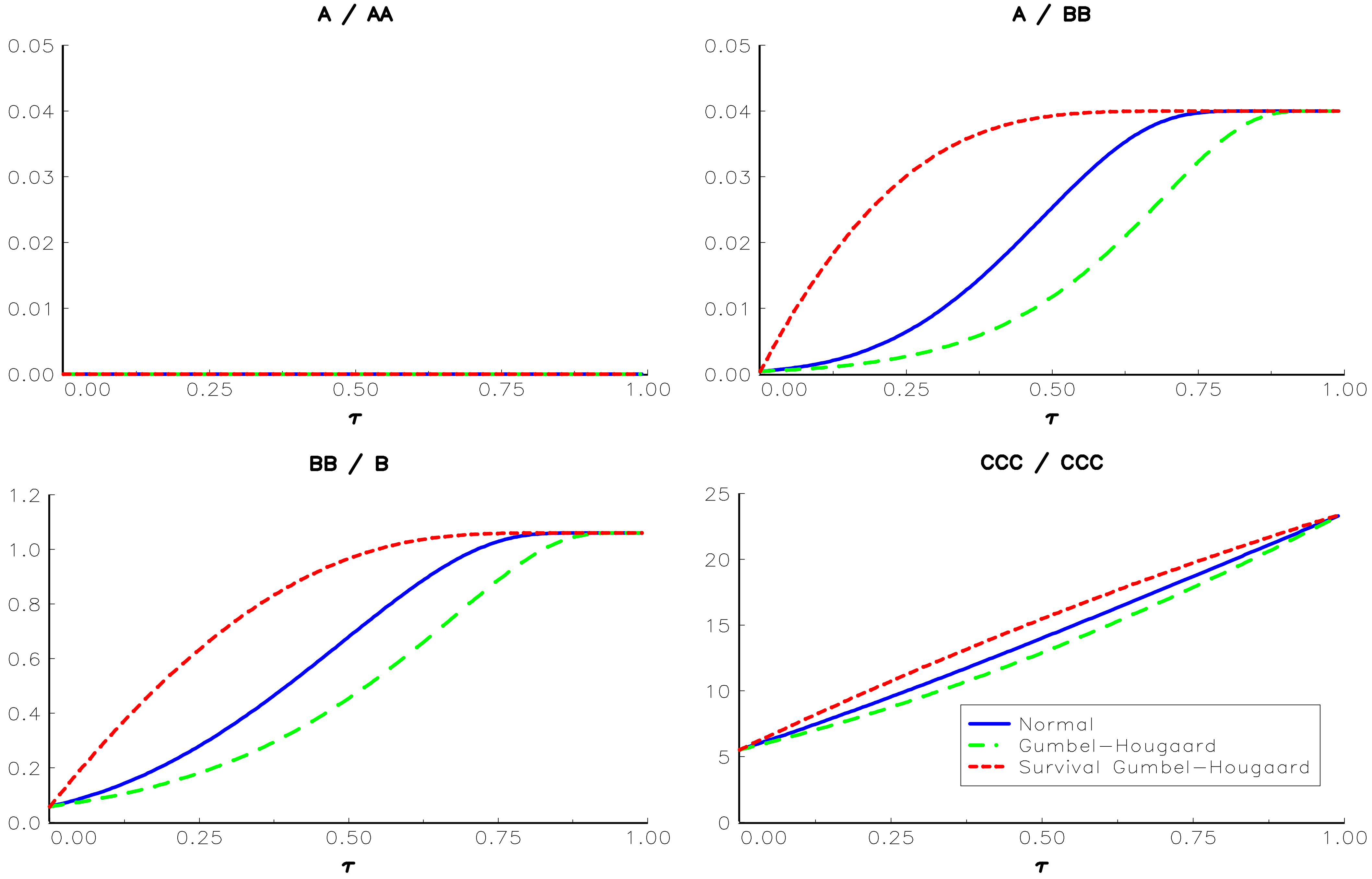


Five−year joint default probability (in %)

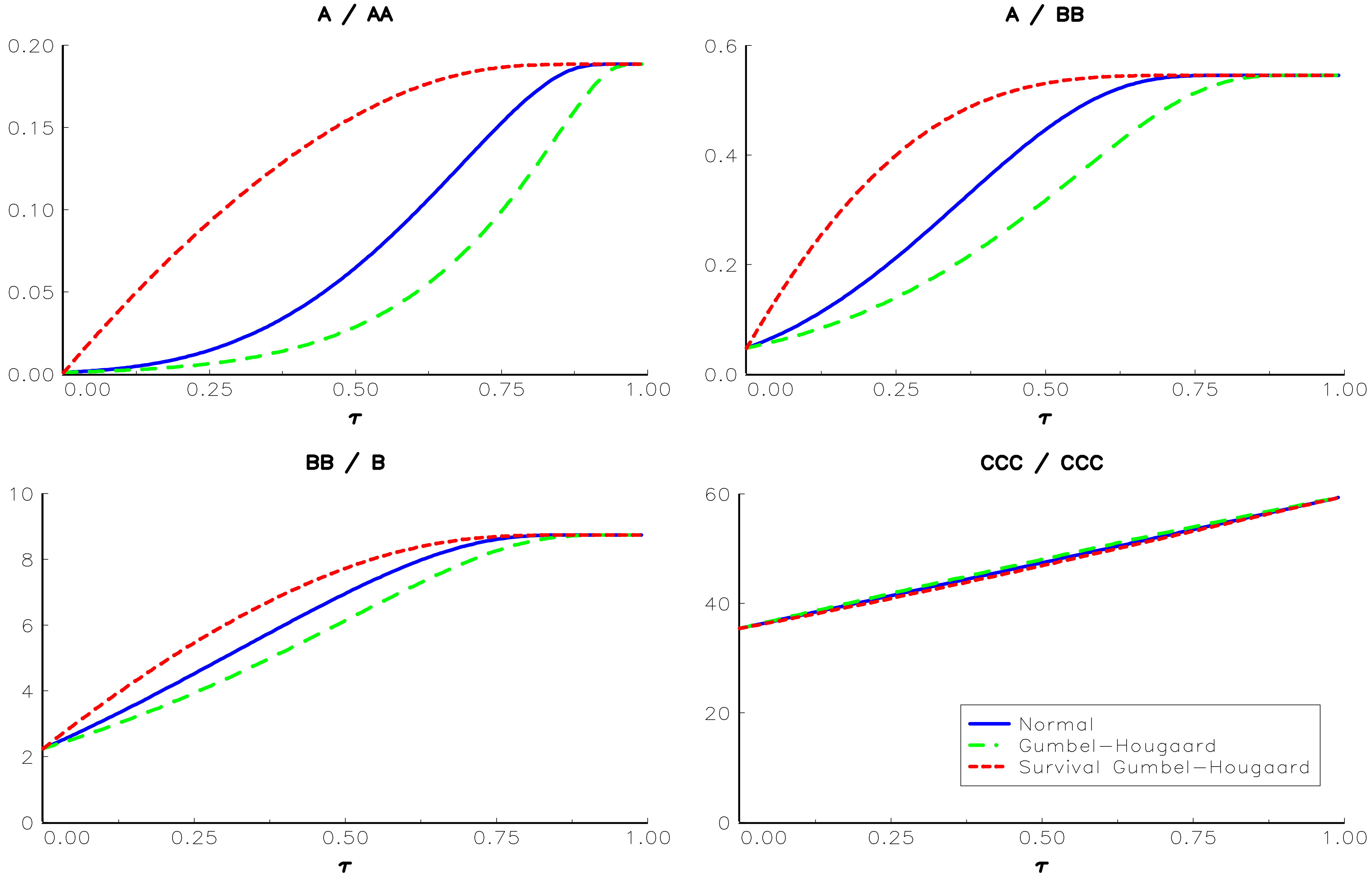

A / AA
A / BB
BB / B
CCC / CCC
0.50
0.25
0.00
−0.25
−0.50
1.0
0.5
0.0
−0.5
−1.0
−1.0
−0.5
0.0
0.5
1.0
ρ
1 year
5 years
10 years
Discrete default correlation $\rho_D(C_1,C_2)$



Lower quantile−dependent measure λ(u)



One−year joint default probability (in %)



Five−year joint default probability (in %)

Computing the risk of a portfolio

Credit risk measure can be obtained by Monte Carlo methods. To simulate the random variables $R_{i_1}, \ldots, R_{i_N}$, we consider the following two step algorithm:

1. we simulate the random numbers $u_1, \ldots, u_N$ from the copula $\mathbf{C}$;
2. the random variates $r_1, \ldots, r_N$ are obtained by the inverse distribution function method:

$$r_n = \pi_i^{(-1)}(u_n)$$

$\Rightarrow$ Credit risk measure = credit value-at-risk with two components: downgrading risk and default risk.

Some problems = credit modelling + recovery rates modelling.

Initial rating	AAA	AA	BBB	BBB	BB	CCC
1	AAA	AA	BBB	BBB	BB	CCC
2	AA	AA	BBB	BBB	BB	CCC
3	AAA	AA	BBB	BBB	BB	D
4	AAA	AA	BBB	A	BB	CCC
5	AAA	AA	A	BBB	BB	CCC
6	AAA	AA	BBB	BBB	BB	CCC
7	AAA	A	BBB	BBB	BB	D
8	AAA	AA	BBB	BBB	BB	BB
9	AAA	AA	BBB	BBB	BB	CCC
10	AAA	AA	BBB	BBB	BB	CCC

Table 6: Simulation of final ratings with $\boldsymbol{\rho}_1$

Initial rating	AAA	AA	BBB	BBB	BB	CCC
1	AAA	AA	BBB	BBB	B	B
2	AA	AA	A	A	BB	CCC
3	AAA	AA	BBB	BBB	BB	D
4	AAA	AA	BBB	A	B	CCC
5	AAA	AA	BBB	BBB	B	CCC
6	AAA	AA	BBB	BBB	BB	CCC
7	AAA	AA	BBB	BBB	BB	D
8	AAA	AA	BB	BBB	BB	B
9	AAA	AA	BBB	BB	BB	D
10	AAA	AA	BBB	BBB	BB	CCC

Table 7: Simulation of final ratings with $\boldsymbol{\rho}_2$

$$\boldsymbol{\rho}_1 = \begin{bmatrix} 1 & 0.25 & 0.75 & 0.5 & 0.25 & 0.2 \\ & 1 & 0.5 & 0.25 & 0.25 & 0.1 \\ & & 1 & 0.75 & 0.25 & 0.2 \\ & & & 1 & 0.25 & 0.5 \\ & & & & 1 & 0.5 \\ & & & & & 1 \end{bmatrix} \quad \boldsymbol{\rho}_2 = \begin{bmatrix} 1 & -0.25 & -0.75 & -0.5 & -0.25 & -0.2 \\ & 1 & 0.5 & 0.25 & 0.25 & 0.1 \\ & & 1 & 0.75 & 0.25 & 0.2 \\ & & & 1 & -0.25 & 0.5 \\ & & & & 1 & -0.5 \\ & & & & & 1 \end{bmatrix}$$

5.1.2 The actuarial approach (CreditRisk+)

More results are in CMMRR [2001]. We just give here some ideas about the copula used by CreditRisk+.

Notations T = time horizon.

B_n = Bernoulli random variable which indicates if the credit n has defaulted.

p_n = stochastic parameter of B_n with

$$p_n = P_n \sum \theta_{n,m} X_m$$

P_n is estimated for each obligator via any credit rating system. X_m are M independent Gamma distributed factors.

Given those factors, the B_n are assumed to be conditionally independent.

One common risk factor

Let us denote $X \sim \Gamma(\alpha, \beta)$, $g(x)$ the pdf of X, $\mathbf{F}(t_1, \ldots, t_N)$ the joint distribution of defaults, $\mathbf{F}_n(t_n)$ the marginal distributions and $\mathbf{F}_n^x(t_n)$ the conditional distribution. We have

$$\mathbf{F}(t_1, \ldots, t_N) = \int_0^\infty \prod_{n=1}^N \mathbf{F}_n^x(t_n)\, g(x)\, \mathrm{d}x$$

It comes that

$$\mathbf{F}(t_1, t_2) = \left(1 + \frac{1}{\alpha}\right) \mathbf{F}_1(t_1)\, \mathbf{F}_2(t_2) - \frac{1}{\alpha}\left(\mathbf{F}_1(t_1) + \mathbf{F}_2(t_2) - 1\right)^+$$

The copula is then

$$\mathbf{C}(u_1, u_2; \alpha) = \left(1 + \frac{1}{\alpha}\right) u_1 u_2 - \frac{1}{\alpha}\left(u_1 + u_2 - 1\right)^+$$

Remark 5 *The previous analysis is not exact. Only the subcopula can be defined for the four continuity points (see CMMRR [2001] for more details).*

Approximation case

We suppose that P_n are small. In this case, we use a Poisson approximation

$$1 - P_n \frac{X}{\alpha\beta} = \exp\left(-P_n \frac{X}{\alpha\beta}\right)$$

We can show that

$$\mathbf{F}\left(t_1, \ldots, t_N\right) = \phi\left(\sum_{n=1}^{N} -\ln \mathbf{F}'_n\left(t_n\right)\right)$$

with

$$\mathbf{F}'_n\left(t_n\right) = \begin{cases} \exp\left(-\frac{P_n}{\alpha\beta}\right) & \text{if} \quad 0 \leq t_n < 1 \\ 1 & \text{if} \quad t_n > 1 \end{cases}$$

ϕ is the Laplace transform associated with $\Gamma\left(\alpha, \beta\right)$. By remarking that

$$-\ln \mathbf{F}'_n\left(t_n\right) = \frac{P_n}{\alpha\beta} \simeq \phi^{-1}\left(\mathbf{F}_n\left(t_n\right)\right)$$

we have

$$\mathbf{F}(t_1,\ldots,t_N) = \phi\left(\sum_{n=1}^{N}\phi^{-1}(\mathbf{F}_n(t_n))\right) \tag{2}$$
$$= \left(\mathbf{F}_1(t_1)^{-\frac{1}{\alpha}}+\ldots+\mathbf{F}_N(t_N)^{-\frac{1}{\alpha}}-N+1\right)^{-\alpha}$$

The dependence function is then the Cook-Johnson copula. Note that equation (2) corresponds to Archimedean copulas. Moreover, the generator of the Archimedean copulas is the inverse of a Laplace transform. In this case, we can give a new probabilistic interpretation, because this model is a frailty model* (see CMMRR [2001] for a discussion on the dependence in CreditRisk+).

⇒ Extensions when the distribution of X is not Gamma can be found in CMMRR [2001].

*This result comes from the fact that the random variables B_n are assumed to be conditionally independent.

5.1.3 The survival approach

see Coutant, Martineu, Messines, Riboulet and Roncalli [2001].

5.2 Pricing of credit derivatives

A default is generally described by a *survival* function $\mathbf{S}(t) = \Pr\{T > t\}$. Let $\breve{\mathbf{C}}$ be a *survival* copula. A multivariate survival distributions $\mathbf{S}$ can be defined as follows

$$\mathbf{S}(t_1, \ldots, t_N) = \breve{\mathbf{C}}(\mathbf{S}_1(t_1), \ldots, \mathbf{S}_N(t_N))$$

where $(\mathbf{S}_1, \ldots, \mathbf{S}_N)$ are the marginal survival functions. Nelsen [1999] notices that "*$\bar{C}$ couples the joint survival function to its univariate margins in a manner completely analogous to the way in which a copula connects the joint distribution function to its margins*".

$\Rightarrow$ Introducing correlation between defaultable securities can then be done using the copula framework (see Li [2000] and Maccarinelli and Maggiolini [2000]).

5.2.1 First-to-Default valuation

Let us define the first-to-default τ as follows

$$\tau = \min\left(T_1, \ldots, T_N\right)$$

Nelsen [1999] shows that the survival function of τ is given by the *diagonal section* of the survival copula.

Let $\mathbf{C}$ be a copula. Its survival copula is given by the following formula

$$\breve{\mathbf{C}}\left(u_1, \ldots, u_N\right) = \bar{\mathbf{C}}\left(1 - u_1, \ldots, 1 - u_n, \ldots, 1 - u_N\right)$$

with

$$\bar{\mathbf{C}}\left(u_1, \ldots, u_n, \ldots, u_N\right) = \sum_{n=0}^{N} \left[(-1)^n \sum_{\mathbf{u} \in \mathcal{Z}(N-n,N)} \mathbf{C}\left(\mathbf{u}\right) \right]$$

where $\mathcal{Z}\left(M, N\right)$ denotes the set $\left\{ \mathbf{u} \in [0,1]^N \mid \sum_{n=1}^{N} \mathcal{X}_{\{1\}}\left(u_n\right) = M \right\}$.

When the copula is radially symmetric, we have

$$\breve{\mathbf{C}} = \mathbf{C}$$

The survival distribution $\mathbf{S}$ of τ is

$$\mathbf{S}(t) = \mathbf{C}(\mathbf{S}_1(t), \ldots, \mathbf{S}_N(t))$$

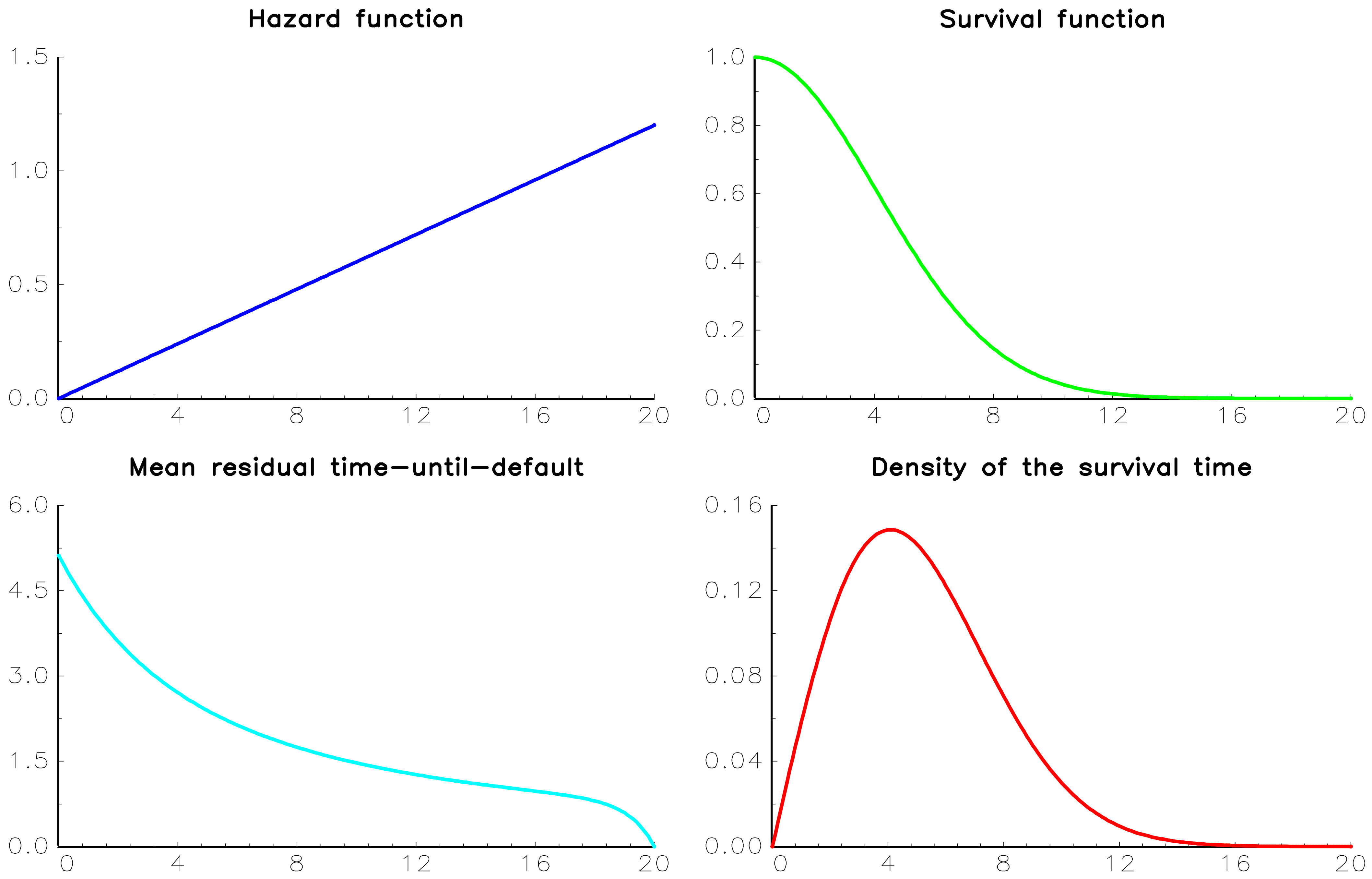
It comes that the density of τ is given by

$$\begin{aligned} f(t) &= -\partial_t \mathbf{S}(t) \\ &= \sum_{n=1}^{N} \partial_n \mathbf{C}(\mathbf{S}_1(t), \ldots, \mathbf{S}_N(t)) \times f_n(t) \end{aligned}$$

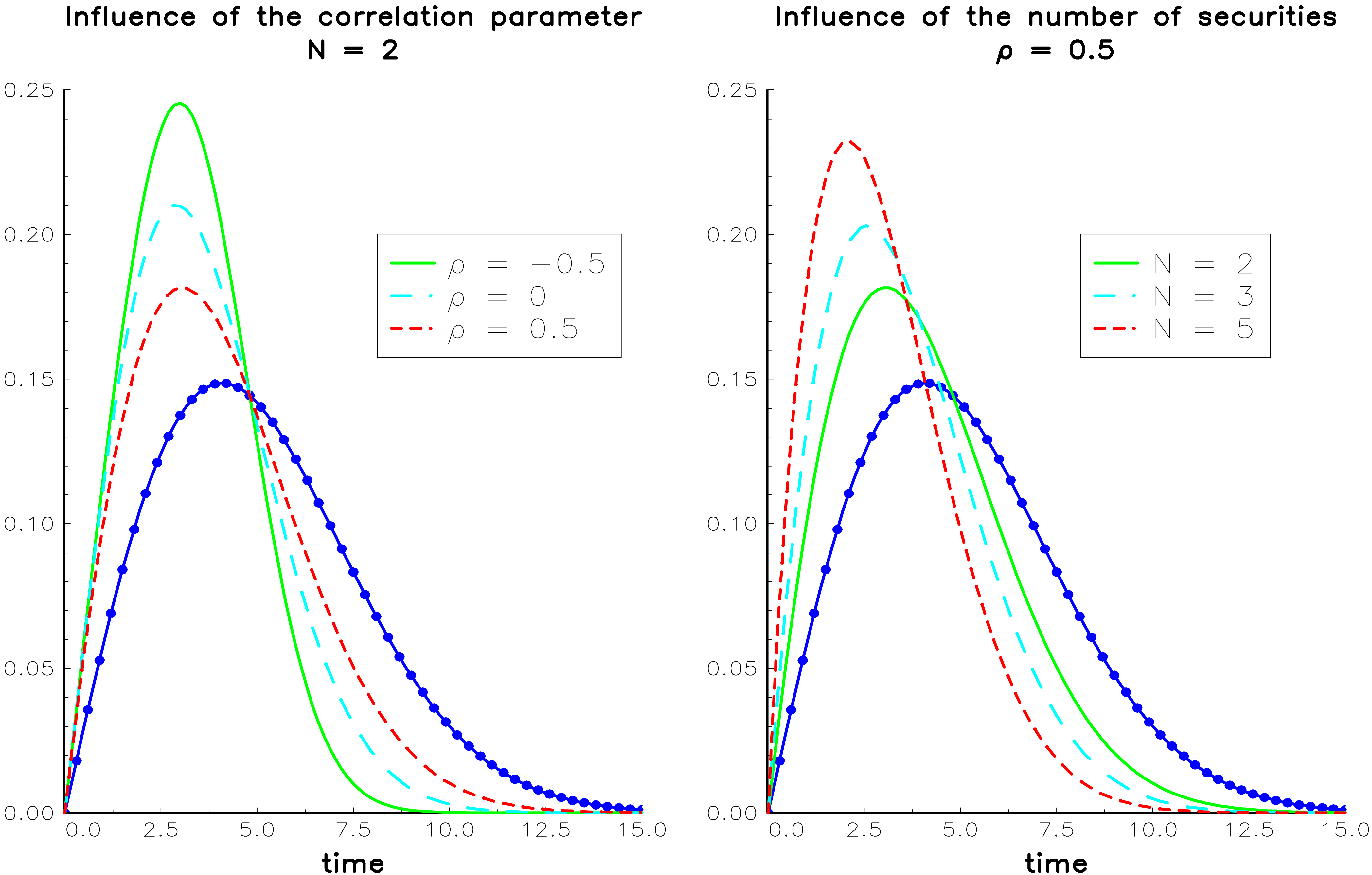
5.2.2 Example

N credit events, default of each credit event given by a Weibull survival function (the baseline hazard is constant and equal to 3% per year and the Weibull parameter is 2).

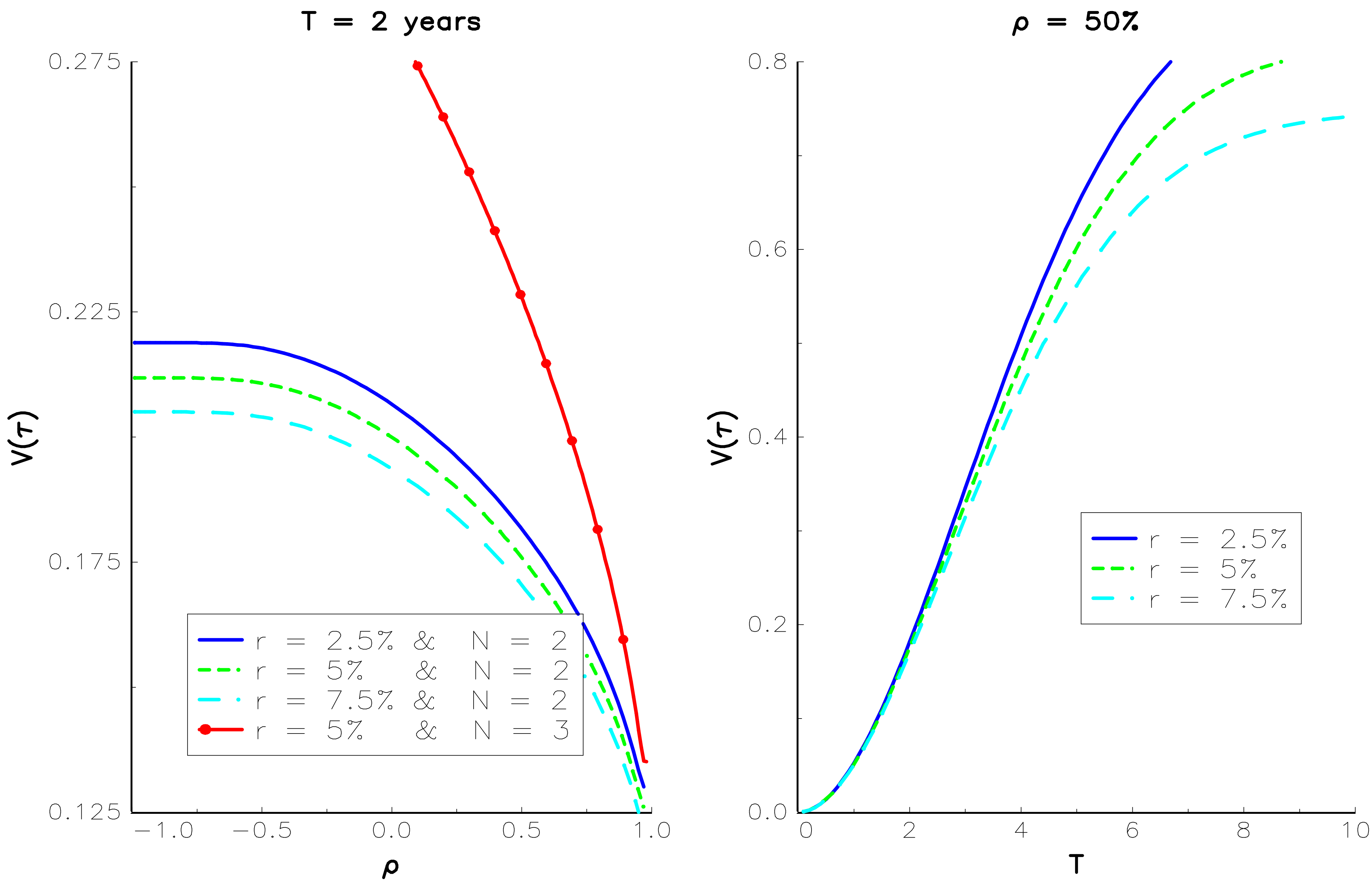
$\mathbf{C}$ is a Normal copula of dimension N = very tractable (N can be very large) and $\partial_n \mathbf{C}$ is a Normal copula of dimension $N-1$.



Weibull survival time



Density of the first-to-default τ



Premium of the first-to-default claim

5.2.3 Other credit derivatives

see Coutant, Martineu, Messines, Riboulet and Roncalli [2001].

6 Conclusion

> *The study of copulas and the role they play in probability, statistics, and stochastic processes is a subject still in its infancy. There are many open problems and much work to be done* (Nelsen [1999], page 4).

In finance, the use of copulas is very recent (Embrechts, McNeil and Straumann [1999]). In one year, great progress have been made. Nevertheless, the finance industry needs more works on copulas and their applications. And there are many research directions to explore. Moreover, many pedagogical works have to be done in order to familiarize the finance industry with copulas.

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